TRIBONACCI IDEAL AND IDEAL-STAR CONVERGENCE IN NON-ARCHIMEDEAN NEUTROSOPHIC NORMED SPACES

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Abstract. This study introduces and investigates valuation-compatible Tribonacci ideal convergence in non-Archimedean neutrosophic normed spaces. The proposed framework combines four essential structures: the truth-membership, indeterminacy-membership, and falsity-membership components of a neutrosophic norm, a non-Archimedean scalar valuation, the matrix domain generated by the Tribonacci sequence, and convergence determined by a proper admissible ideal. Several equivalent characterizations of Tribonacci $\mathcal{J}$-convergence are obtained in terms of exceptional index sets, the filter associated with the ideal, and componentwise ideal limits. The uniqueness of the Tribonacci $\mathcal{J}$-limit is established, and the method is shown to be preserved under addition and scalar multiplication. We further introduce Tribonacci $\mathcal{J}^*$-convergence and prove its equivalence with Tribonacci $\mathcal{J}$-convergence whenever the ideal satisfies the additive property (AP). Tribonacci $\mathcal{J}$-Cauchy sequences are also defined, and a complete Cauchy characterization is derived under suitable completeness and ideal assumptions. In addition, preservation results for neutrosophically continuous linear operators, inclusion relations between convergence classes generated by nested ideals, and sufficient conditions for valuation-compatible regularity of the Tribonacci matrix are presented. Illustrative examples demonstrate that the proposed convergence method is strictly more general than ordinary Tribonacci convergence. The explicit role of the non-Archimedean valuation distinguishes this approach from classical real- and complex-valued Tribonacci summability methods.

1. Introduction

The study of generalized convergence occupies a central position in modern analysis, summability theory, sequence spaces, approximation theory, and operator theory. Classical convergence requires all terms of a sequence, except possibly finitely many, to lie arbitrarily close to the proposed limit. Although this condition is natural and powerful, it may be too restrictive in problems involving sparse perturbations, irregular observations, matrix transformations, or uncertainty. For this reason, several alternative convergence methods have been developed in which exceptional indices are allowed, provided that they are sufficiently small in an appropriate mathematical sense.

The mathematical analysis of uncertainty began with the theory of fuzzy sets introduced by Zadeh [1]. This theory replaced the binary membership rule of an ordinary set by a membership function taking values in the unit interval. The underlying ideas subsequently motivated the development of fuzzy vector spaces [3], fuzzy metric spaces [4, 6], and fuzzy normed linear spaces [5]. Important structural and topological developments in fuzzy metric and fuzzy normed spaces were established by Grabiec [7], Fang [8], Bag and Samanta [9], and Nădăban

2020 *Mathematics Subject Classification.* 40A35; 40C05; 40J05; 46S10; 46S40.

Key words and phrases. Tribonacci matrix; ideal convergence; $\mathcal{J}^*$-convergence; non-Archimedean neutrosophic normed space; admissible ideal; Tribonacci ideal-Cauchy sequence; matrix regularity; continuous linear operator.

[10, 11]. These contributions showed that many fundamental notions of functional analysis, including convergence, continuity, boundedness, completeness, and fixed points, admit meaningful extensions to fuzzy environments.

Atanassov introduced intuitionistic fuzzy sets as a refinement of the classical fuzzy-set model [2]. In an intuitionistic fuzzy set, membership and non-membership are specified separately, while their remaining part represents hesitation. Park introduced intuitionistic fuzzy metric spaces [12], and Saadati and Park investigated their associated topological structures [13]. These developments provided a suitable basis for studying generalized convergence in spaces whose geometry is described simultaneously through membership and non-membership functions.

Statistical convergence is one of the most influential alternatives to ordinary convergence. It was introduced independently in the early work of Fast [29] and was later systematically developed by Fridy [30]. Connor established important relations between statistical convergence and strong Cesàro summability [31], while Fridy studied statistical limit points [32]. Miller obtained a measure-theoretic subsequence characterization of statistical convergence [33]. The concept was subsequently extended to double sequences by Mursaleen and Edely [34]. These studies demonstrated that statistical convergence effectively describes sequences whose nonconvergent behavior is restricted to an index set of natural density zero.

Statistical convergence has also been investigated in fuzzy and intuitionistic fuzzy settings. Karakuş, Demirci, and Duman introduced statistical convergence in intuitionistic fuzzy normed spaces [14]. Mursaleen and Mohiuddine studied lacunary statistical convergence and statistical convergence of double sequences in these spaces [16, 17]. Mohiuddine and Lohani considered a generalized form of statistical convergence [15], while Kumar and Mursaleen examined (λ, μ)-statistical convergence of double sequences [20]. Related extensions involving sequences of functions and generalized summability methods were obtained by Karakaya et al. [21] and Savaş and Gürdal [22]. These results indicate that statistical convergence is compatible with the membership-based geometry of fuzzy and intuitionistic fuzzy normed spaces.

A more general method is obtained by replacing natural density with an ideal of subsets of $\mathbb{N}$. The systematic theory of ideal convergence was developed by Kostyrko, Šalát, and Wilczyński [35]. Given a proper admissible ideal $\mathcal{I}$, a sequence is $\mathcal{I}$-convergent when the set of indices at which the sequence fails to satisfy a prescribed neighborhood condition belongs to $\mathcal{I}$. Thus, the ideal determines which sets of indices are regarded as negligible. This framework includes ordinary convergence, statistical convergence, and several summability-based methods as particular cases.

The closely related notion of $\mathcal{I}^*$-convergence requires ordinary convergence along a subsequence whose index set belongs to the filter associated with $\mathcal{I}$. The relation between $\mathcal{I}$- and $\mathcal{I}^*$-convergence in topological spaces was investigated by Lahiri and Das [36]. Balcerzak, Dems, and Komisarski studied statistical and ideal convergence for sequences of functions [37], whereas Komisarski examined pointwise ideal convergence and convergence in measure [38]. Mrożek established an ideal analogue of Egorov's theorem [39]. Das and Ghosal studied $\mathcal{I}$-Cauchy sequences and the additive property (AP) [40], while Das, Savaş, and Ghosal developed ideal-based generalizations of summability methods [41]. More recently, Connor and Leonetti obtained a characterization of matrices that preserve convergence with respect to two ideals [42]. These results show that ideals provide a flexible set-theoretic mechanism for controlling exceptional behavior.

Ideal convergence has also been combined with intuitionistic fuzzy structures. Mursaleen, Mohiuddine, and Edely studied ideal convergence of double sequences in intuitionistic fuzzy normed spaces [18]. Debnath introduced lacunary ideal convergence in intuitionistic fuzzy normed linear spaces [19], and Konwar and Debnath investigated $\mathcal{I}_\lambda$-convergence in intuitionistic fuzzy n-normed spaces [23]. These investigations demonstrate that ideal methods can be

adapted effectively to spaces whose convergence is determined by several uncertainty parameters.

The neutrosophic framework provides a further generalization by using three components representing truth, indeterminacy, and falsity. In contrast to an intuitionistic fuzzy model, these components may be treated as distinct sources of information. Kirişci and Şimşek introduced neutrosophic metric spaces [25] and developed neutrosophic normed spaces together with statistical convergence [26]. Ersan subsequently examined ρ-statistical convergence in neutrosophic normed spaces [28]. Recent developments have substantially broadened convergence theory in neutrosophic settings. Deferred statistical summability in neutrosophic n-normed spaces was investigated in [65], while completeness and Cartesian-product structures for neutrosophic rectangular n-normed spaces were studied in [66]. Rough and deferred $\mathcal{I}$-statistical convergence methods were further developed in [64, 69]. A broader account of recent advances in neutrosophic analysis, algebra, graph theory, and their applications is presented in [67, 68]. These studies demonstrate that neutrosophic norms form a natural setting in which convergence can simultaneously measure the growth of truth-membership and the decay of indeterminacy- and falsity-membership.

Non-Archimedean analysis supplies another important component of the present framework. A non-Archimedean valuation satisfies the strong triangle inequality

$$|\alpha + \beta| \leq \max\{|\alpha|, |\beta|\},$$

and the corresponding normed spaces possess an ultrametric geometry. This geometry appears naturally in p-adic analysis and in various hierarchical mathematical models. Statistical convergence in intuitionistic fuzzy normed spaces over non-Archimedean fields was studied by Saranya and Suja [27]. The combination of non-Archimedean geometry with a neutrosophic norm is therefore particularly suitable for studying sequences whose uncertainty has three components and whose underlying distance satisfies a strong triangle inequality.

Another major line of research concerns matrix transformations and matrix domains of sequence spaces. Freedman, Sember, and Raphael studied Cesàro-type summability spaces generated by matrix methods [43]. Kirişci and Başar constructed sequence spaces through the domain of a generalized difference matrix [44], while Kara investigated topological and geometrical properties of sequence spaces associated with Fibonacci numbers [45]. Matrix-domain techniques are useful because they permit a convergence method to be encoded into a lower triangular matrix and allow the transformed sequence to be studied through the geometry of an appropriate sequence space.

The Tribonacci sequence is generated by a third-order recurrence and was discussed in the early work of Feinberg [46]. Its use in summability and sequence-space theory extends the corresponding Fibonacci constructions by incorporating three consecutive terms of the recurrence. Dağlı and Yaying developed paranormed sequence spaces generated by a regular Tribonacci matrix [47]. In the related Fibonacci setting, Kirişci introduced Fibonacci statistical convergence in intuitionistic fuzzy normed spaces [24], and Kirişci and Karaisa obtained Korovkin-type approximation results through Fibonacci statistical convergence [48]. These developments indicate that recurrence-generated triangular matrices provide effective tools for defining new sequence spaces and generalized convergence methods.

A direct combination of ideal convergence, Tribonacci transformations, and non-Archimedean intuitionistic fuzzy norms was recently considered by Ansari [49]. That work developed Tribonacci ideal convergence and Tribonacci ideal-Cauchy sequences in a non-Archimedean intuitionistic fuzzy normed space. The intuitionistic fuzzy framework, however, is based on membership and non-membership components. A neutrosophic normed space involves three separate functions, and consequently each convergence condition must control truth-membership,

indeterminacy-membership, and falsity-membership simultaneously. This produces a different analytical structure and requires separate proofs for the associated exceptional sets, filter characterizations, Cauchy conditions, and preservation properties.

To bridge this gap, the present paper develops Tribonacci ideal convergence in non-Archimedean neutrosophic normed spaces. Let

$$(\mathcal{U}, \Omega, \Upsilon, \Theta, *, \diamond)$$

be a non-Archimedean neutrosophic normed space and let $\mathcal{T} = (\tau_{nk})$ be the Tribonacci matrix. For a sequence $x = (x_k)$ in $\mathcal{U}$, the transformed sequence is defined by

$$\mathcal{T}_n(x) = \sum_{k=1}^{n} \tau_{nk} x_k.$$

The proposed convergence method requires the exceptional sets associated with the three neutrosophic components to belong to a proper admissible ideal $\mathcal{I}$.

The main contributions of this paper can be summarized as follows:

(1) We introduce Tribonacci $\mathcal{I}$-convergence in a non-Archimedean neutrosophic normed space and derive equivalent formulations in terms of componentwise exceptional sets, the associated filter, and ideal limits of the three neutrosophic components.
(2) We prove the uniqueness of the Tribonacci $\mathcal{I}$-limit and establish the stability of the convergence method under addition and scalar multiplication.
(3) We introduce Tribonacci $\mathcal{I}^*$-convergence and show that it implies Tribonacci $\mathcal{I}$-convergence. Under property (AP), the converse implication is obtained, thereby establishing the equivalence of the two methods for an important class of ideals.
(4) We define Tribonacci $\mathcal{I}$-Cauchy sequences and prove that every Tribonacci $\mathcal{I}$-convergent sequence is Tribonacci $\mathcal{I}$-Cauchy. In a complete space and under property (AP), the corresponding Cauchy criterion is established.
(5) We study preservation under neutrosophically continuous linear operators and derive inclusion relations for convergence methods generated by nested ideals.
(6) We provide examples that distinguish ordinary Tribonacci convergence from Tribonacci ideal convergence and illustrate the roles of the ideal, its associated filter, and the non-Archimedean neutrosophic norm.

The remainder of the paper is organized as follows. Section 2 recalls the basic concepts of ideals, filters, continuous t-norms and t-conorms, non-Archimedean valuations, neutrosophic norms, and the Tribonacci matrix. Section 3 introduces Tribonacci $\mathcal{I}$-convergence and establishes its fundamental properties. Section 4 investigates Tribonacci $\mathcal{I}^*$-convergence and the additive property (AP). Section 5 develops Tribonacci $\mathcal{I}$-Cauchy sequences and the corresponding completeness results. Section 6 studies preservation under linear operators and inclusion relations between ideals. Illustrative examples and concluding remarks are presented in the final section.

2. Preliminaries

This section recalls the basic concepts concerning ideals and filters, non-Archimedean valued spaces, neutrosophic norms, and the Tribonacci matrix transformation that will be used throughout the paper. Unless otherwise stated, $\mathbb{N} = \{1, 2, 3, \ldots\}$ and $\mathbb{K}$ denotes a non-Archimedean valued field of characteristic zero.

2.1. Ideals and Associated Filters.

We begin with the notions of an ideal and its associated filter.

Definition 2.1. *A nonempty family $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is called an* ideal *on $\mathbb{N}$ if the following conditions are satisfied:*

(1) *if $A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$;*

(2) *if* $A, B \in \mathcal{I}$, *then* $A \cup B \in \mathcal{I}$.

The ideal $\mathcal{I}$ *is called:*

(1) proper *if* $\mathbb{N} \notin \mathcal{I}$;
(2) admissible *if* $\{n\} \in \mathcal{I}$ *for every* $n \in \mathbb{N}$.

Every proper ideal determines a filter in a natural manner.

Definition 2.2. *Let* $\mathcal{I}$ *be a proper ideal on* $\mathbb{N}$. *The family*

$$\mathcal{F}(\mathcal{I}) = \{A \subseteq \mathbb{N} : A^c \in \mathcal{I}\},$$

where

$$A^c = \mathbb{N} \setminus A,$$

is called the filter associated with $\mathcal{I}$.

The family $\mathcal{F}(\mathcal{I})$ satisfies the following properties:

$$\varnothing \notin \mathcal{F}(\mathcal{I}), \qquad \mathbb{N} \in \mathcal{F}(\mathcal{I}),$$

and it is closed under finite intersections and supersets.

Example 2.1. *The family*

$$\mathcal{I}_d = \{A \subseteq \mathbb{N} : d(A) = 0\},$$

where

$$d(A) = \lim_{n\to\infty} \frac{1}{n} |A \cap \{1, 2, \ldots, n\}|,$$

whenever the limit exists, is a proper admissible ideal. It is called the ideal of subsets of $\mathbb{N}$ *having natural density zero.*

2.2. **Non-Archimedean Neutrosophic Normed Spaces.** We next recall the necessary concepts related to non-Archimedean valued fields and neutrosophic normed spaces.

Definition 2.3. *A mapping* $|\cdot| : \mathbb{K} \longrightarrow [0, \infty)$ *is called a* non-Archimedean valuation *if, for all* $\alpha, \beta \in \mathbb{K}$,

(1) $|\alpha| = 0$ *if and only if* $\alpha = 0$;
(2) $|\alpha\beta| = |\alpha||\beta|$;
(3) $|\alpha + \beta| \leq \max\{|\alpha|, |\beta|\}$.

The pair $(\mathbb{K}, |\cdot|)$ *is then called a* non-Archimedean valued field.

Definition 2.4. *Let* $\mathcal{U}$ *be a vector space over* $\mathbb{K}$. *A mapping* $\|\cdot\| : \mathcal{U} \longrightarrow [0, \infty)$ *is called a* non-Archimedean norm *if, for all* $u, v \in \mathcal{U}$ *and* $\alpha \in \mathbb{K}$,

(1) $\|u\| = 0$ *if and only if* $u = 0$;
(2) $\|\alpha u\| = |\alpha|\|u\|$;
(3) $\|u + v\| \leq \max\{\|u\|, \|v\|\}$.

Example 2.2. *Let* $\mathbb{K} = \mathbb{Q}_p$ *be the field of p-adic numbers equipped with the p-adic absolute value* $|\cdot|_p$, *and let* $\mathcal{U} = \mathbb{Q}_p^m$. *For* $u = (u_1, u_2, \ldots, u_m) \in \mathcal{U}$, *define as*

$$\|u\|_p = \max_{1\leq i\leq m} |u_i|_p.$$

Then, $\|\cdot\|_p$ *is a non-Archimedean norm on* $\mathcal{U}$.

Indeed, $\|u\|_p = 0$ *if and only if* $|u_i|_p = 0$ *for every* i, *which is equivalent to* $u = 0$. *Moreover, for every* $\alpha \in \mathbb{Q}_p$,

$$\begin{aligned}\|\alpha u\|_p &= \max_{1\leq i\leq m} |\alpha u_i|_p \\ &= |\alpha|_p \max_{1\leq i\leq m} |u_i|_p \\ &= |\alpha|_p \|u\|_p.\end{aligned}$$

Finally, for $u, v \in \mathcal{U}$, the strong triangle inequality of the p-adic absolute value gives

$$|u_i + v_i|_p \leq \max\{|u_i|_p, |v_i|_p\}.$$

Consequently,

$$\begin{aligned}\|u+v\|_p &= \max_{1\leq i\leq m} |u_i + v_i|_p \\ &\leq \max\left\{\max_{1\leq i\leq m} |u_i|_p, \max_{1\leq i\leq m} |v_i|_p\right\} \\ &= \max\{\|u\|_p, \|v\|_p\}.\end{aligned}$$

Therefore, $\left(\mathbb{Q}_p^m, \|\cdot\|_p\right)$ is a non-Archimedean normed space.

Example 2.3. *For $a, b \in [0,1]$, define $a * b = \min\{a, b\}$ and $a \diamond b = \max\{a, b\}$. Then, $*$ is a continuous t-norm and $\diamond$ is a continuous t-conorm. Indeed, both operations are associative, commutative, continuous, and nondecreasing in each variable. Moreover,*

$$a * 1 = \min\{a, 1\} = a \text{ and } a \diamond 0 = \max\{a, 0\} = a.$$

The standard examples used in this paper are

$$a * b = \min\{a, b\} \text{ and } a \diamond b = \max\{a, b\}.$$

Definition 2.5. *Let $*$ be a continuous t-norm and let $\diamond$ be a continuous t-conorm. Three mappings*

$$\mathsf{P}, \mathsf{Q}, \mathsf{R} : X \times (0, \infty) \longrightarrow [0, 1]$$

are said to define a non-Archimedean neutrosophic norm on X if, for all $x, y \in X$, $s, t > 0$, and $\alpha \in \mathbb{K} \setminus \{0\}$, the following conditions hold:

(1) $\mathsf{P}(x, t) = 1$ *if and only if* $x = 0$;
(2) $\mathsf{Q}(x, t) = 0$ *if and only if* $x = 0$;
(3) $\mathsf{R}(x, t) = 0$ *if and only if* $x = 0$;
(4)
$$\mathsf{P}(\alpha x, t) = \mathsf{P}\left(x, \frac{t}{|\alpha|}\right);$$
(5)
$$\mathsf{Q}(\alpha x, t) = \mathsf{Q}\left(x, \frac{t}{|\alpha|}\right);$$
(6)
$$\mathsf{R}(\alpha x, t) = \mathsf{R}\left(x, \frac{t}{|\alpha|}\right);$$
(7)
$$\mathsf{P}(x + y, \max\{s, t\}) \geq \mathsf{P}(x, s) * \mathsf{P}(y, t);$$
(8)
$$\mathsf{Q}(x + y, \max\{s, t\}) \leq \mathsf{Q}(x, s) \diamond \mathsf{Q}(y, t);$$
(9)
$$\mathsf{R}(x + y, \max\{s, t\}) \leq \mathsf{R}(x, s) \diamond \mathsf{R}(y, t);$$
(10) $\mathsf{P}(x, \cdot)$ *is continuous and nondecreasing, whereas* $\mathsf{Q}(x, \cdot)$ *and* $\mathsf{R}(x, \cdot)$ *are continuous and nonincreasing;*
(11)
$$\lim_{t\to\infty} \mathsf{P}(x, t) = 1, \qquad \lim_{t\to\infty} \mathsf{Q}(x, t) = 0, \qquad \lim_{t\to\infty} \mathsf{R}(x, t) = 0.$$

The system $(X, \mathsf{P}, \mathsf{Q}, \mathsf{R}, *, \diamond)$ is called a non-Archimedean neutrosophic normed space, abbreviated as an NA-NNS.

Example 2.4. *Let $\mathbb{K} = \mathbb{Q}_p$ be the field of p-adic numbers and let $X = \mathbb{Q}_p^m$ be equipped with the non-Archimedean norm*

$$\|x\|_p = \max_{1\le i\le m} |x_i|_p, \qquad x = (x_1, \dots, x_m) \in X.$$

Define by

$$a * b = \min\{a, b\}, \qquad a \diamond b = \max\{a, b\},$$

and, for $x \in X$ and $t > 0$, the set

$$\mathsf{P}(x,t) = \exp\left(-\frac{\|x\|_p}{t}\right),$$

$$\mathsf{Q}(x,t) = 1 - \exp\left(-\frac{\|x\|_p}{t}\right)$$

and

$$\mathsf{R}(x,t) = 1 - \exp\left(-\frac{2\|x\|_p}{t}\right).$$

Clearly,

$$\mathsf{P}(x,t) = 1 \quad \Longleftrightarrow \quad x = 0,$$

and

$$\mathsf{Q}(x,t) = \mathsf{R}(x,t) = 0 \quad \Longleftrightarrow \quad x = 0.$$

Moreover, for every $\alpha \in \mathbb{Q}_p \setminus \{0\}$,

$$\mathsf{P}(\alpha x, t) = \mathsf{P}\left(x, \frac{t}{|\alpha|_p}\right),$$

with analogous identities for Q and R.

Since

$$\|x + y\|_p \le \max\{\|x\|_p, \|y\|_p\},$$

we obtain

$$\mathsf{P}(x + y, \max\{s, t\}) \ge \min\{\mathsf{P}(x, s), \mathsf{P}(y, t)\},$$
$$\mathsf{Q}(x + y, \max\{s, t\}) \le \max\{\mathsf{Q}(x, s), \mathsf{Q}(y, t)\},$$

and

$$\mathsf{R}(x + y, \max\{s, t\}) \le \max\{\mathsf{R}(x, s), \mathsf{R}(y, t)\}.$$

The functions P, Q, and R have the required continuity and monotonicity properties, and

$$\lim_{t\to\infty} \mathsf{P}(x,t) = 1, \qquad \lim_{t\to\infty} \mathsf{Q}(x,t) = 0, \qquad \lim_{t\to\infty} \mathsf{R}(x,t) = 0.$$

Therefore, $\left(\mathbb{Q}_p^m, \mathsf{P}, \mathsf{Q}, \mathsf{R}, \min, \max\right)$ is a non-Archimedean neutrosophic normed space.

2.3. **Convergence, Cauchy Sequences, and Completeness.** We now recall the ordinary notions of convergence and completeness in an NA-NNS.

Definition 2.6. *A sequence (x_n) in an NA-NNS $(X, \mathsf{P}, \mathsf{Q}, \mathsf{R}, *, \diamond)$ is said to converge to $x \in X$ if, for every $\varepsilon \in (0, 1)$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that*

$$\mathsf{P}(x_n - x, t) > 1 - \varepsilon,$$

$$\mathsf{Q}(x_n - x, t) < \varepsilon$$

and

$$\mathsf{R}(x_n - x, t) < \varepsilon$$

for every $n \ge n_0$.

We write

$$(\mathsf{P}, \mathsf{Q}, \mathsf{R})\text{-}\lim_{n\to\infty} x_n = x.$$

Definition 2.7. *A sequence (x_n) is called a Cauchy sequence in the NA-NNS if, for every $\varepsilon \in (0,1)$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that, for all $m, n \geq n_0$,*

$$\mathsf{P}(x_n - x_m, t) > 1 - \varepsilon,$$

$$\mathsf{Q}(x_n - x_m, t) < \varepsilon$$

and

$$\mathsf{R}(x_n - x_m, t) < \varepsilon.$$

The space is called complete if every Cauchy sequence is convergent.

Definition 2.8. *An NA-NNS is said to be* complete *if every Cauchy sequence in the space converges to an element of the space.*

2.4. **The Tribonacci Matrix Transformation.** Let (t_n) denote the Tribonacci sequence defined by

$$t_0 = 0, \qquad t_1 = t_2 = 1, \qquad t_3 = 2,$$

and

$$t_n = t_{n-1} + t_{n-2} + t_{n-3}, \qquad n \geq 4.$$

The first few terms are

$$0, 1, 1, 2, 4, 7, 13, 24, \ldots.$$

Definition 2.9. *The Tribonacci matrix $\mathcal{T} = (\tau_{nk})$ is the lower triangular matrix whose entries are defined by*

$$\tau_{nk} = \begin{cases} \dfrac{2t_k}{t_{n+2} + t_{n-1}}, & 1 \leq k \leq n, \\ 0, & k > n. \end{cases}$$

For a sequence $x = (x_k)$ in $\mathcal{U}$, its Tribonacci transform is the sequence $\mathcal{T}x = (\mathcal{T}_n(x))$ defined by

$$\mathcal{T}_n(x) = \sum_{k=1}^{n} \tau_{nk} x_k = \sum_{k=1}^{n} \frac{2t_k}{t_{n+2} + t_{n-1}} x_k.$$

Because $\mathcal{T}$ is lower triangular, the nth transformed term $\mathcal{T}_n(x)$ depends only on the first n terms $x_1, x_2, \ldots, x_n$ of the original sequence. The concepts introduced above provide the necessary framework for studying Tribonacci ideal convergence, Tribonacci $\mathcal{I}^*$-convergence, and Tribonacci ideal-Cauchy sequences in non-Archimedean neutrosophic normed spaces.

3. Basic Properties of Tribonacci Ideal Convergence

In this section, we introduce Tribonacci ideal convergence in a non-Archimedean neutrosophic normed space and establish some of its fundamental properties. In particular, we present equivalent characterizations, prove the uniqueness of the Tribonacci $\mathcal{J}$-limit, and show that this convergence method is preserved under addition and scalar multiplication.

Definition 3.1. *A sequence $x = (x_k)$ in an NA-NNS is said to be Tribonacci $\mathcal{J}$-convergent to $\ell \in X$ if, for every $\varepsilon \in (0,1)$ and $t > 0$, the set*

$$A_{\mathbb{T}}(\varepsilon, t; \ell) = \left\{ n \in \mathbb{N} : \begin{array}{l} \mathsf{P}(\mathbb{T}_n(x) - \ell, t) \leq 1 - \varepsilon \\ \text{or } \mathsf{Q}(\mathbb{T}_n(x) - \ell, t) \geq \varepsilon \\ \text{or } \mathsf{R}(\mathbb{T}_n(x) - \ell, t) \geq \varepsilon \end{array} \right\}$$

belongs to $\mathcal{J}$.

In this case, we write

$$\mathcal{J}_{(\mathsf{P},\mathsf{Q},\mathsf{R})}\text{-}\lim_{n\to\infty} \mathbb{T}_n(x) = \ell.$$

Example 3.1. *Let $(X, \|\cdot\|)$ be a non-Archimedean normed space over a non-Archimedean valued field $\mathbb{K}$. Define*

$$\mathsf{P}(x,t) = \frac{t}{t+\|x\|}, \qquad \mathsf{Q}(x,t) = \frac{\|x\|}{t+\|x\|}$$

and

$$\mathsf{R}(x,t) = \frac{\|x\|}{t+\|x\|}, \qquad x \in X, \quad t > 0.$$

Let

$$a * b = \min\{a,b\}, \qquad a \diamond b = \max\{a,b\}.$$

Then, $(X, \mathsf{P}, \mathsf{Q}, \mathsf{R}, *, \diamond)$ is a non-Archimedean neutrosophic normed space.

Consider the admissible ideal

$$\mathcal{J}_d = \{A \subseteq \mathbb{N} : d(A) = 0\},$$

where

$$d(A) = \lim_{n\to\infty} \frac{1}{n} |A \cap \{1,2,\ldots,n\}|$$

denotes the natural density of A, whenever this limit exists.

Let $u \in X$ be a fixed nonzero vector, and define the sequence (z_n) by

$$z_n = \begin{cases} u, & n = m^2 \text{ for some } m \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

Since the Tribonacci matrix $\mathbb{T} = (a_{nk})$ is lower triangular and $a_{nn} \neq 0$ for every $n \in \mathbb{N}$, there exists a unique sequence $x = (x_k)$ in X such that

$$\mathbb{T}_n(x) = z_n$$

for every $n \in \mathbb{N}$. Indeed, the sequence x may be defined recursively by

$$x_1 = a_{11}^{-1} z_1$$

and

$$x_n = a_{nn}^{-1}\left(z_n - \sum_{k=1}^{n-1} a_{nk} x_k\right), \qquad n \geq 2.$$

Now, we show that x is Tribonacci $\mathcal{J}_d$-convergent to zero. Let $\varepsilon \in (0,1)$ and $t > 0$. Consider the exceptional set

$$A_{\mathbb{T}}(\varepsilon, t; 0) = \left\{ n \in \mathbb{N} : \begin{array}{c} \mathsf{P}(\mathbb{T}_n(x), t) \leq 1-\varepsilon, \\ \text{or } \mathsf{Q}(\mathbb{T}_n(x), t) \geq \varepsilon, \\ \text{or } \mathsf{R}(\mathbb{T}_n(x), t) \geq \varepsilon \end{array} \right\}.$$

If n is not a perfect square, then

$$\mathbb{T}_n(x) = z_n = 0.$$

Therefore,

$$\mathsf{P}(\mathbb{T}_n(x), t) = \mathsf{P}(0, t) = 1,$$
$$\mathsf{Q}(\mathbb{T}_n(x), t) = \mathsf{Q}(0, t) = 0$$

and

$$\mathsf{R}(\mathbb{T}_n(x), t) = \mathsf{R}(0, t) = 0.$$

Thus, no non-square index belongs to the exceptional set. Hence,

$$A_{\mathbb{T}}(\varepsilon, t; 0) \subseteq S,$$

where

$$S = \{m^2 : m \in \mathbb{N}\}.$$

The set of perfect squares has natural density zero because

$$\frac{|S \cap \{1, 2, \dots, n\}|}{n} = \frac{\lfloor \sqrt{n} \rfloor}{n} \longrightarrow 0.$$

Consequently,

$$S \in \mathcal{J}_d.$$

Since $\mathcal{J}_d$ is hereditary, we obtain

$$A_{\mathbb{T}}(\varepsilon, t; 0) \in \mathcal{J}_d.$$

Since $\varepsilon \in (0,1)$ and $t > 0$ were arbitrary, it follows that

$$\mathcal{J}_{d,(\mathsf{P},\mathsf{Q},\mathsf{R})}\text{-}\lim_{n\to\infty} \mathbb{T}_n(x) = 0.$$

However, $(\mathbb{T}_n(x))$ does not converge ordinarily to zero. Indeed, fix $t_0 > 0$ and put

$$c = \frac{\|u\|}{t_0 + \|u\|} > 0.$$

Choose

$$\varepsilon_0 = \frac{c}{2}.$$

For every perfect square $n = m^2$, we have

$$\mathbb{T}_{m^2}(x) = u,$$

and therefore

$$\mathsf{P}(u, t_0) = 1 - c \leq 1 - \varepsilon_0,$$
$$\mathsf{Q}(u, t_0) = c \geq \varepsilon_0$$

and

$$\mathsf{R}(u, t_0) = c \geq \varepsilon_0.$$

Since there are infinitely many perfect squares, the ordinary convergence conditions fail infinitely often.

Thus, x is Tribonacci $\mathcal{J}_d$-convergent to zero, although it is not ordinarily Tribonacci-convergent to zero.

The following theorem gives several equivalent formulations of Tribonacci $\mathcal{J}$-convergence.

Theorem 3.1. *Let $x = (x_k)$ be a sequence in an NA-NNS and let $\ell \in X$. The following assertions are equivalent:*

(1) *x is Tribonacci $\mathcal{J}$-convergent to ℓ;*

(2) *for every $\varepsilon \in (0,1)$ and $t > 0$, each of the sets*

$$A_{\mathsf{P}} = \{n \in \mathbb{N} : \mathsf{P}(\mathbb{T}_n(x) - \ell, t) \leq 1 - \varepsilon\},$$
$$A_{\mathsf{Q}} = \{n \in \mathbb{N} : \mathsf{Q}(\mathbb{T}_n(x) - \ell, t) \geq \varepsilon\}$$

and

$$A_{\mathsf{R}} = \{n \in \mathbb{N} : \mathsf{R}(\mathbb{T}_n(x) - \ell, t) \geq \varepsilon\}$$

belongs to $\mathcal{J}$;

(3) *for every $\varepsilon \in (0,1)$ and $t > 0$, the set*

$$G_{\mathbb{T}}(\varepsilon, t; \ell) = \left\{ n \in \mathbb{N} : \begin{array}{l} \mathsf{P}(\mathbb{T}_n(x) - \ell, t) > 1 - \varepsilon, \\ \mathsf{Q}(\mathbb{T}_n(x) - \ell, t) < \varepsilon, \\ \mathsf{R}(\mathbb{T}_n(x) - \ell, t) < \varepsilon \end{array} \right\}$$

belongs to $\mathcal{F}(\mathcal{J})$;

(4) *for every* $t > 0$,

$$\mathcal{J}\text{-}\lim_{n\to\infty} \mathsf{P}(\mathbb{T}_n(x) - \ell, t) = 1,$$
$$\mathcal{J}\text{-}\lim_{n\to\infty} \mathsf{Q}(\mathbb{T}_n(x) - \ell, t) = 0$$

and

$$\mathcal{J}\text{-}\lim_{n\to\infty} \mathsf{R}(\mathbb{T}_n(x) - \ell, t) = 0.$$

Proof. The exceptional set in Definition 3.1 is exactly

$$A_{\mathbb{T}}(\varepsilon, t; \ell) = A_{\mathsf{P}} \cup A_{\mathsf{Q}} \cup A_{\mathsf{R}}.$$

Since an ideal is closed under finite unions and under taking subsets, assertions (1) and (2) are equivalent.

Moreover,

$$G_{\mathbb{T}}(\varepsilon, t; \ell) = \mathbb{N} \setminus A_{\mathbb{T}}(\varepsilon, t; \ell).$$

Therefore,

$$A_{\mathbb{T}}(\varepsilon, t; \ell) \in \mathcal{J}$$

if and only if

$$G_{\mathbb{T}}(\varepsilon, t; \ell) \in \mathcal{F}(\mathcal{J}).$$

Thus, (1) and (3) are equivalent.

Finally, assertion (4) is precisely the componentwise formulation of assertion (2), using the definition of ideal convergence for real sequences. Hence all the assertions are equivalent. □

We next establish the uniqueness of the Tribonacci $\mathcal{J}$-limit.

Theorem 3.2. *The Tribonacci $\mathcal{J}$-limit of a sequence in an NA-NNS is unique.*

Proof. Suppose that

$$\mathcal{J}_{(\mathsf{P},\mathsf{Q},\mathsf{R})}\text{-}\lim_{n\to\infty} \mathbb{T}_n(x) = \ell_1$$

and

$$\mathcal{J}_{(\mathsf{P},\mathsf{Q},\mathsf{R})}\text{-}\lim_{n\to\infty} \mathbb{T}_n(x) = \ell_2.$$

Let $\varepsilon \in (0, 1)$. By the continuity of $*$ and $\diamond$, there exists $\delta \in (0, 1)$ such that

$$(1 - \delta) * (1 - \delta) > 1 - \varepsilon$$

and

$$\delta \diamond \delta < \varepsilon.$$

For any fixed $t > 0$, define

$$G_i = G_{\mathbb{T}}(\delta, t; \ell_i), \qquad i = 1, 2.$$

Where

$$G_i = \left\{ n \in \mathbb{N} : \begin{array}{c} \mathsf{P}(\mathbb{T}_n(x) - \ell_i, t) > 1 - \delta, \\ \mathsf{Q}(\mathbb{T}_n(x) - \ell_i, t) < \delta, \\ \mathsf{R}(\mathbb{T}_n(x) - \ell_i, t) < \delta \end{array} \right\}.$$

By Theorem 3.1, we have

$$G_1, G_2 \in \mathcal{F}(\mathcal{J}).$$

Consequently,

$$G_1 \cap G_2 \in \mathcal{F}(\mathcal{J}),$$

and hence $G_1 \cap G_2 \neq \varnothing$.

Choose $n \in G_1 \cap G_2$. Then,

$$\mathsf{P}(\mathbb{T}_n(x) - \ell_i, t) > 1 - \delta,$$
$$\mathsf{Q}(\mathbb{T}_n(x) - \ell_i, t) < \delta$$

and

$$\mathsf{R}(\mathbb{T}_n(x) - \ell_i, t) < \delta$$

for $i = 1, 2$.

By the homogeneity properties of the non-Archimedean neutrosophic norm and the fact that $|-1| = 1$, we have

$$\mathsf{P}(-u, t) = \mathsf{P}\left(u, \frac{t}{|-1|}\right) = \mathsf{P}(u, t),$$
$$\mathsf{Q}(-u, t) = \mathsf{Q}(u, t)$$

and

$$\mathsf{R}(-u, t) = \mathsf{R}(u, t)$$

for every $u \in X$. Therefore,

$$\mathsf{P}(\ell_1 - \mathbb{T}_{n_0}(x), t) > 1 - \delta,$$
$$\mathsf{Q}(\ell_1 - \mathbb{T}_{n_0}(x), t) < \delta$$

and

$$\mathsf{R}(\ell_1 - \mathbb{T}_{n_0}(x), t) < \delta.$$

Now, we observe that

$$\ell_1 - \ell_2 = \big(\ell_1 - \mathbb{T}_{n_0}(x)\big) + \big(\mathbb{T}_{n_0}(x) - \ell_2\big).$$

Applying the non-Archimedean triangle property for the truth-membership function, with $\max\{t, t\} = t$, we obtain

$$\begin{aligned}
\mathsf{P}(\ell_1 - \ell_2, t) &= \mathsf{P}\left(\ell_1 - \mathbb{T}_n(x) + \mathbb{T}_n(x) - \ell_2, t\right) \\
&\geq \mathsf{P}(\ell_1 - \mathbb{T}_n(x), t) * \mathsf{P}(\mathbb{T}_n(x) - \ell_2, t) \\
&> (1 - \delta) * (1 - \delta) \\
&> 1 - \varepsilon.
\end{aligned}$$

Similarly, by the non-Archimedean triangle property for the indeterminacy-membership function, we get

$$\begin{aligned}
\mathsf{Q}(\ell_1 - \ell_2, t) &\leq \mathsf{Q}(\ell_1 - \mathbb{T}_n(x), t) \diamond \mathsf{Q}(\mathbb{T}_n(x) - \ell_2, t) \\
&< \delta \diamond \delta \\
&< \varepsilon,
\end{aligned}$$

In the same manner, using the corresponding property of the falsity-membership function, we have

$$\begin{aligned}
\mathsf{R}(\ell_1 - \ell_2, t) &\leq \mathsf{R}(\ell_1 - \mathbb{T}_n(x), t) \diamond \mathsf{R}(\mathbb{T}_n(x) - \ell_2, t) \\
&< \delta \diamond \delta \\
&< \varepsilon,
\end{aligned}$$

Thus, for every $\varepsilon \in (0, 1)$,

$$\mathsf{P}(\ell_1 - \ell_2, t) > 1 - \varepsilon,$$
$$\mathsf{Q}(\ell_1 - \ell_2, t) < \varepsilon$$

and

$$\mathsf{R}(\ell_1 - \ell_2, t) < \varepsilon.$$

Since ε is arbitrary, it follows that

$$\mathsf{P}(\ell_1 - \ell_2, t) = 1, \qquad \mathsf{Q}(\ell_1 - \ell_2, t) = 0$$

and

$$\mathsf{R}(\ell_1 - \ell_2, t) = 0.$$

Therefore, $\ell_1 - \ell_2 = 0$, and hence $\ell_1 = \ell_2$. Hence, the Tribonacci $\mathcal{J}$-limit of a sequence in an NA-NNS is unique. □

The following result shows that Tribonacci $\mathcal{J}$-convergence is compatible with the linear structure of the underlying space.

Theorem 3.3. *Let $x = (x_k)$ and $y = (y_k)$ be sequences in an NA-NNS. Suppose that*

$$\mathcal{J}_{(\mathsf{P},\mathsf{Q},\mathsf{R})}\text{-}\lim_{n\to\infty} \mathbb{T}_n(x) = \ell_1 \text{ and } \mathcal{J}_{(\mathsf{P},\mathsf{Q},\mathsf{R})}\text{-}\lim_{n\to\infty} \mathbb{T}_n(y) = \ell_2.$$

Then:

(1)

$$\mathcal{J}_{(\mathsf{P},\mathsf{Q},\mathsf{R})}\text{-}\lim_{n\to\infty} \mathbb{T}_n(x+y) = \ell_1 + \ell_2;$$

(2) *for every $\alpha \in \mathbb{K}$,*

$$\mathcal{J}_{(\mathsf{P},\mathsf{Q},\mathsf{R})}\text{-}\lim_{n\to\infty} \mathbb{T}_n(\alpha x) = \alpha\ell_1.$$

Proof. We prove the two assertions separately.

Part (1). We first prove that

$$\mathbb{T}_n(x+y)$$

is Tribonacci $\mathcal{J}$-convergent to $\ell_1 + \ell_2$.

Let $\varepsilon \in (0,1)$ and $t > 0$. Choose $\delta \in (0,1)$ such that

$$(1-\delta) * (1-\delta) > 1 - \varepsilon$$

and

$$\delta \diamond \delta < \varepsilon.$$

Define

$$G_x = \left\{ n \in \mathbb{N} : \begin{array}{l} \mathsf{P}(\mathbb{T}_n(x) - \ell_1, t) > 1-\delta, \\ \mathsf{Q}(\mathbb{T}_n(x) - \ell_1, t) < \delta, \\ \mathsf{R}(\mathbb{T}_n(x) - \ell_1, t) < \delta \end{array} \right\}$$

and

$$G_y = \left\{ n \in \mathbb{N} : \begin{array}{l} \mathsf{P}(\mathbb{T}_n(y) - \ell_2, t) > 1-\delta, \\ \mathsf{Q}(\mathbb{T}_n(y) - \ell_2, t) < \delta, \\ \mathsf{R}(\mathbb{T}_n(y) - \ell_2, t) < \delta \end{array} \right\}.$$

By the assumed Tribonacci $\mathcal{J}$-convergence of x and y, and by Theorem 3.1, we have

$$G_x, G_y \in \mathcal{F}(\mathcal{J}).$$

Since $\mathcal{F}(\mathcal{J})$ is a filter, it is closed under finite intersections. Therefore,

$$G_x \cap G_y \in \mathcal{F}(\mathcal{J}).$$

Let $n \in G_x \cap G_y$. By the linearity of the Tribonacci transformation,

$$\begin{aligned} \mathbb{T}_n(x+y) &= \sum_{k=1}^{n} a_{nk}(x_k + y_k) \\ &= \sum_{k=1}^{n} a_{nk}x_k + \sum_{k=1}^{n} a_{nk}y_k \\ &= \mathbb{T}_n(x) + \mathbb{T}_n(y). \end{aligned}$$

Consequently,

$$\begin{aligned} \mathbb{T}_n(x+y) - (\ell_1 + \ell_2) &= \mathbb{T}_n(x) + \mathbb{T}_n(y) - \ell_1 - \ell_2 \\ &= \big(\mathbb{T}_n(x) - \ell_1\big) + \big(\mathbb{T}_n(y) - \ell_2\big). \end{aligned}$$

Using the non-Archimedean triangle property for the truth-membership function, together with $\max\{t,t\} = t$, we obtain

$$\begin{aligned}
&\mathsf{P}\big(\mathbb{T}_n(x+y) - (\ell_1+\ell_2), t\big) \\
&\quad = \mathsf{P}\left(\big(\mathbb{T}_n(x) - \ell_1\big) + \big(\mathbb{T}_n(y) - \ell_2\big), \max\{t,t\}\right) \\
&\quad \geq \mathsf{P}\big(\mathbb{T}_n(x) - \ell_1, t\big) * \mathsf{P}\big(\mathbb{T}_n(y) - \ell_2, t\big) \\
&\quad > (1-\delta) * (1-\delta) \\
&\quad > 1 - \varepsilon.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\mathsf{Q}\big(\mathbb{T}_n(x+y) - (\ell_1+\ell_2), t\big) \\
&\quad \leq \mathsf{Q}\big(\mathbb{T}_n(x) - \ell_1, t\big) \diamond \mathsf{Q}\big(\mathbb{T}_n(y) - \ell_2, t\big) \\
&\quad < \delta \diamond \delta \\
&\quad < \varepsilon,
\end{aligned}$$

and

$$\begin{aligned}
&\mathsf{R}\big(\mathbb{T}_n(x+y) - (\ell_1+\ell_2), t\big) \\
&\quad \leq \mathsf{R}\big(\mathbb{T}_n(x) - \ell_1, t\big) \diamond \mathsf{R}\big(\mathbb{T}_n(y) - \ell_2, t\big) \\
&\quad < \delta \diamond \delta \\
&\quad < \varepsilon.
\end{aligned}$$

Therefore, every $n \in G_x \cap G_y$ belongs to the good-index set

$$G_{x+y} = \left\{ n \in \mathbb{N} : \begin{array}{l} \mathsf{P}\big(\mathbb{T}_n(x+y) - (\ell_1+\ell_2), t\big) > 1-\varepsilon, \\ \mathsf{Q}\big(\mathbb{T}_n(x+y) - (\ell_1+\ell_2), t\big) < \varepsilon, \\ \mathsf{R}\big(\mathbb{T}_n(x+y) - (\ell_1+\ell_2), t\big) < \varepsilon \end{array} \right\}.$$

Thus,

$$G_x \cap G_y \subseteq G_{x+y}.$$

Since $G_x \cap G_y \in \mathcal{F}(\mathcal{J})$ and a filter is upward closed, it follows that

$$G_{x+y} \in \mathcal{F}(\mathcal{J}).$$

Equivalently,

$$\mathbb{N} \setminus G_{x+y} \in \mathcal{J}.$$

By Theorem 3.1, this means that

$$\mathcal{J}_{(\mathsf{P},\mathsf{Q},\mathsf{R})}\text{-}\lim_{n\to\infty} \mathbb{T}_n(x+y) = \ell_1 + \ell_2.$$

This proves part (1).

Part (2). Let $\alpha \in \mathbb{K}$. We prove that $\mathbb{T}_n(\alpha x)$ is Tribonacci $\mathcal{J}$-convergent to $\alpha\ell_1$. If $\alpha = 0$, then

$$\mathbb{T}_n(\alpha x) = \mathbb{T}_n(0) = 0$$

for every $n \in \mathbb{N}$, and

$$\alpha\ell_1 = 0.$$

Therefore,

$$\mathbb{T}_n(\alpha x) - \alpha\ell_1 = 0.$$

By the definiteness properties of the neutrosophic norm,

$$\mathsf{P}(0,t) = 1, \qquad \mathsf{Q}(0,t) = 0$$

and

$$\mathsf{R}(0,t) = 0.$$

Hence, the exceptional set is empty and therefore belongs to $\mathcal{J}$. Thus, the conclusion is immediate when $\alpha = 0$.

Now, we suppose that $\alpha \neq 0$. Let $\varepsilon \in (0,1)$ and $t > 0$ be arbitrary. Since $|\alpha| > 0$, the number

$$\frac{t}{|\alpha|}$$

is well defined and positive.

Define

$$G_\alpha = \left\{ n \in \mathbb{N} : \begin{array}{r} \mathsf{P}\left(\mathbb{T}_n(x) - \ell_1, \dfrac{t}{|\alpha|}\right) > 1-\varepsilon, \\ \mathsf{Q}\left(\mathbb{T}_n(x) - \ell_1, \dfrac{t}{|\alpha|}\right) < \varepsilon, \\ \mathsf{R}\left(\mathbb{T}_n(x) - \ell_1, \dfrac{t}{|\alpha|}\right) < \varepsilon \end{array} \right\}.$$

Since

$$\mathcal{J}_{(\mathsf{P},\mathsf{Q},\mathsf{R})}\text{-}\lim_{n\to\infty} \mathbb{T}_n(x) = \ell_1,$$

Theorem 3.1 gives

$$G_\alpha \in \mathcal{F}(\mathcal{J}).$$

By the linearity of the Tribonacci transformation,

$$\begin{aligned} \mathbb{T}_n(\alpha x) &= \sum_{k=1}^{n} a_{nk}(\alpha x_k) \\ &= \alpha \sum_{k=1}^{n} a_{nk} x_k \\ &= \alpha \mathbb{T}_n(x). \end{aligned}$$

Therefore,

$$\begin{aligned} \mathbb{T}_n(\alpha x) - \alpha\ell_1 &= \alpha \mathbb{T}_n(x) - \alpha\ell_1 \\ &= \alpha\big(\mathbb{T}_n(x) - \ell_1\big). \end{aligned}$$

Let $n \in G_\alpha$. By the homogeneity property of P,

$$\begin{aligned} &\mathsf{P}\big(\mathbb{T}_n(\alpha x) - \alpha\ell_1, t\big) \\ &\quad = \mathsf{P}\left(\alpha\big(\mathbb{T}_n(x) - \ell_1\big), t\right) \\ &\quad = \mathsf{P}\left(\mathbb{T}_n(x) - \ell_1, \frac{t}{|\alpha|}\right) \\ &\quad > 1 - \varepsilon. \end{aligned}$$

Similarly,

$$\begin{aligned} &\mathsf{Q}\big(\mathbb{T}_n(\alpha x) - \alpha\ell_1, t\big) \\ &\quad = \mathsf{Q}\left(\mathbb{T}_n(x) - \ell_1, \frac{t}{|\alpha|}\right) \\ &\quad < \varepsilon \end{aligned}$$

and

$$\begin{aligned} &\mathsf{R}\big(\mathbb{T}_n(\alpha x) - \alpha\ell_1, t\big) \\ &\quad = \mathsf{R}\left(\mathbb{T}_n(x) - \ell_1, \frac{t}{|\alpha|}\right) \\ &\quad < \varepsilon. \end{aligned}$$

Consequently,

$$G_\alpha \subseteq \left\{ n \in \mathbb{N} : \begin{array}{c} \mathsf{P}\big(\mathbb{T}_n(\alpha x) - \alpha\ell_1, t\big) > 1-\varepsilon, \\ \mathsf{Q}\big(\mathbb{T}_n(\alpha x) - \alpha\ell_1, t\big) < \varepsilon, \\ \mathsf{R}\big(\mathbb{T}_n(\alpha x) - \alpha\ell_1, t\big) < \varepsilon \end{array} \right\}.$$

Since $G_\alpha \in \mathcal{F}(\mathcal{J})$ and the filter is upward closed, the set on the right-hand side also belongs to $\mathcal{F}(\mathcal{J})$. By Theorem 3.1, we conclude that

$$\mathcal{J}_{(\mathsf{P},\mathsf{Q},\mathsf{R})}\text{-}\lim_{n\to\infty} \mathbb{T}_n(\alpha x) = \alpha\ell_1.$$

This completes the proof. □

Corollary 3.1. *Under the hypotheses of Theorem 3.3, for every $\alpha, \beta \in \mathbb{K}$,*

$$\mathcal{J}_{(\mathsf{P},\mathsf{Q},\mathsf{R})}\text{-}\lim_{n\to\infty} \mathbb{T}_n(\alpha x + \beta y) = \alpha\ell_1 + \beta\ell_2.$$

Proof. The conclusion follows directly from the two parts of Theorem 3.3. □

4. Equivalence of Tribonacci $\mathcal{J}$- and $\mathcal{J}^*$-Convergence

In this section, we examine the relationship between Tribonacci $\mathcal{J}$-convergence and Tribonacci $\mathcal{J}^*$-convergence in a non-Archimedean neutrosophic normed space. In general, Tribonacci $\mathcal{J}^*$-convergence implies Tribonacci $\mathcal{J}$-convergence. Under the additional assumption that the ideal satisfies the additive property (AP), the converse also holds.

Definition 4.1. *A sequence $x = (x_k)$ is said to be Tribonacci $\mathcal{J}^*$-convergent to $\ell \in X$ if there exists*

$$M = \{m_1 < m_2 < \cdots\} \in \mathcal{F}(\mathcal{J})$$

such that

$$(\mathsf{P},\mathsf{Q},\mathsf{R})\text{-}\lim_{j\to\infty} \mathbb{T}_{m_j}(x) = \ell.$$

Example 4.1. *Let*

$$(X, \mathsf{P}, \mathsf{Q}, \mathsf{R}, *, \diamond)$$

be a non-Archimedean neutrosophic normed space, and let $u \in X$ be a fixed nonzero vector.

Denote the sets of odd and even natural numbers by

$$\mathcal{O} = \{2j-1 : j \in \mathbb{N}\}$$

and

$$\mathcal{E} = \{2j : j \in \mathbb{N}\},$$

respectively. Define by

$$\mathcal{J}_\mathcal{O} = \{A \subseteq \mathbb{N} : A \cap \mathcal{O} \text{ is finite}\}.$$

Then, $\mathcal{J}_\mathcal{O}$ is a proper admissible ideal of $\mathbb{N}$. Indeed, it contains the empty set and every finite subset of $\mathbb{N}$, and it is closed under taking subsets and finite unions. Moreover,

$$\mathbb{N} \notin \mathcal{J}_\mathcal{O},$$

because

$$\mathbb{N} \cap \mathcal{O} = \mathcal{O}$$

is infinite.

Define a sequence $z = (z_n)$ in X as

$$z_n = \begin{cases} 0, & n \in \mathcal{O}, \\ u, & n \in \mathcal{E}. \end{cases}$$

Assume that the Tribonacci matrix $\mathbb{T} = (a_{nk})$ is lower triangular with

$$a_{nn} \neq 0$$

for every $n \in \mathbb{N}$. Then there exists a unique sequence $x = (x_k)$ in X such that

$$\mathbb{T}_n(x) = z_n$$

for every $n \in \mathbb{N}$. The sequence x can be constructed recursively by

$$x_1 = a_{11}^{-1} z_1$$

and

$$x_n = a_{nn}^{-1} \left(z_n - \sum_{k=1}^{n-1} a_{nk} x_k \right), \qquad n \geq 2.$$

Let

$$M = \mathcal{O} = \{1, 3, 5, \ldots\}.$$

Since

$$\mathbb{N} \setminus M = \mathcal{E}$$

and

$$\mathcal{E} \cap \mathcal{O} = \varnothing,$$

we have

$$\mathcal{E} \in \mathcal{J}_{\mathcal{O}}.$$

Consequently,

$$M \in \mathcal{F}(\mathcal{J}_{\mathcal{O}}).$$

Write as

$$M = \{m_1 < m_2 < \cdots\}, \qquad m_j = 2j - 1.$$

For every $j \in \mathbb{N}$,

$$\mathbb{T}_{m_j}(x) = z_{m_j} = 0.$$

Therefore, for every $\varepsilon \in (0, 1)$ and $t > 0$,

$$\mathsf{P}\big(\mathbb{T}_{m_j}(x) - 0, t\big) = \mathsf{P}(0, t) = 1,$$

$$\mathsf{Q}\big(\mathbb{T}_{m_j}(x) - 0, t\big) = \mathsf{Q}(0, t) = 0$$

and

$$\mathsf{R}\big(\mathbb{T}_{m_j}(x) - 0, t\big) = \mathsf{R}(0, t) = 0.$$

Hence,

$$(\mathsf{P}, \mathsf{Q}, \mathsf{R})\text{-}\lim_{j \to \infty} \mathbb{T}_{m_j}(x) = 0.$$

Because

$$M \in \mathcal{F}(\mathcal{J}_{\mathcal{O}}),$$

it follows from the definition of Tribonacci $\mathcal{J}^*$-convergence that

$$\mathcal{J}^*_{\mathcal{O},(\mathsf{P},\mathsf{Q},\mathsf{R})}\text{-}\lim_{n \to \infty} \mathbb{T}_n(x) = 0.$$

However, the entire transformed sequence does not converge ordinarily to zero. Indeed,

$$\mathbb{T}_{2j}(x) = u$$

for every $j \in \mathbb{N}$. Since $u \neq 0$, the definiteness axioms give, for every fixed $t > 0$,

$$\mathsf{P}(u, t) < 1, \qquad \mathsf{Q}(u, t) > 0$$

and

$$\mathsf{R}(u, t) > 0.$$

Thus, the even-indexed subsequence remains equal to the nonzero vector u, and consequently

$$\mathbb{T}_n(x) \not\longrightarrow 0$$

in the ordinary neutrosophic sense.

Therefore, x is Tribonacci $\mathcal{J}_{\mathcal{O}}^{*}$-convergent to zero, although it is not ordinarily Tribonacci-convergent to zero.

Theorem 4.1. *Every Tribonacci $\mathcal{J}^{*}$-convergent sequence is Tribonacci $\mathcal{J}$-convergent to the same limit.*

Proof. Assume that $x = (x_k)$ is Tribonacci $\mathcal{J}^{*}$-convergent to ℓ. By the definition of Tribonacci $\mathcal{J}^{*}$-convergence, there exists an infinite set

$$M = \{m_1 < m_2 < \cdots\} \in \mathcal{F}(\mathcal{J})$$

such that the subsequence of Tribonacci transforms

$$\big(\mathbb{T}_{m_j}(x)\big)$$

converges ordinarily to ℓ with respect to the non-Archimedean neutrosophic norm. That is,

$$(\mathsf{P}, \mathsf{Q}, \mathsf{R})\text{-}\lim_{j\to\infty} \mathbb{T}_{m_j}(x) = \ell.$$

Now, we prove that x is Tribonacci $\mathcal{J}$-convergent to ℓ.

Let $\varepsilon \in (0,1)$ and $t > 0$ be arbitrary. Since

$$\mathbb{T}_{m_j}(x) \longrightarrow \ell$$

in the neutrosophic norm, there exists an integer $j_0 \in \mathbb{N}$ such that, for every $j \geq j_0$,

$$\mathsf{P}\big(\mathbb{T}_{m_j}(x) - \ell, t\big) > 1 - \varepsilon,$$

$$\mathsf{Q}\big(\mathbb{T}_{m_j}(x) - \ell, t\big) < \varepsilon$$

and

$$\mathsf{R}\big(\mathbb{T}_{m_j}(x) - \ell, t\big) < \varepsilon.$$

Consider the exceptional set

$$A_{\mathbb{T}}(\varepsilon, t; \ell) = \left\{ n \in \mathbb{N} : \begin{array}{c} \mathsf{P}(\mathbb{T}_n(x) - \ell, t) \leq 1 - \varepsilon, \\ \text{or } \mathsf{Q}(\mathbb{T}_n(x) - \ell, t) \geq \varepsilon, \\ \text{or } \mathsf{R}(\mathbb{T}_n(x) - \ell, t) \geq \varepsilon \end{array} \right\}.$$

Now, we claim that

$$A_{\mathbb{T}}(\varepsilon, t; \ell) \subseteq (\mathbb{N} \setminus M) \cup \{m_1, m_2, \ldots, m_{j_0 - 1}\}.$$

Indeed, let

$$n \in A_{\mathbb{T}}(\varepsilon, t; \ell).$$

Suppose that

$$n \notin \mathbb{N} \setminus M.$$

Then, $n \in M$, and hence there exists some $j \in \mathbb{N}$ such that

$$n = m_j.$$

If $j \geq j_0$, then, by the ordinary convergence of the subsequence,

$$\mathsf{P}(\mathbb{T}_{m_j}(x) - \ell, t) > 1 - \varepsilon,$$

$$\mathsf{Q}(\mathbb{T}_{m_j}(x) - \ell, t) < \varepsilon$$

and

$$\mathsf{R}(\mathbb{T}_{m_j}(x) - \ell, t) < \varepsilon.$$

imply that

$$m_j \notin A_{\mathbb{T}}(\varepsilon, t; \ell),$$

which contradicts the assumption that

$$n = m_j \in A_{\mathbb{T}}(\varepsilon, t; \ell).$$

Therefore, if an exceptional index belongs to M, it must be one of the finitely many indices

$$m_1, m_2, \ldots, m_{j_0-1}.$$

This proves the claimed inclusion.

Since

$$M \in \mathcal{F}(\mathcal{J}),$$

the definition of the associated filter gives

$$\mathbb{N} \setminus M \in \mathcal{J}.$$

Furthermore, because $\mathcal{J}$ is admissible, every singleton belongs to $\mathcal{J}$. Thus,

$$\{m_1\}, \{m_2\}, \ldots, \{m_{j_0-1}\} \in \mathcal{J}.$$

Since an ideal is closed under finite unions, it follows that

$$\{m_1, m_2, \ldots, m_{j_0-1}\} \in \mathcal{J}.$$

Hence,

$$(\mathbb{N} \setminus M) \cup \{m_1, m_2, \ldots, m_{j_0-1}\} \in \mathcal{J}.$$

Because an ideal is hereditary, every subset of a member of $\mathcal{J}$ also belongs to $\mathcal{J}$. Therefore, from

$$A_{\mathbb{T}}(\varepsilon, t; \ell) \subseteq (\mathbb{N} \setminus M) \cup \{m_1, m_2, \ldots, m_{j_0-1}\},$$

we get

$$A_{\mathbb{T}}(\varepsilon, t; \ell) \in \mathcal{J}.$$

Since $\varepsilon \in (0, 1)$ and $t > 0$ were arbitrary, the definition of Tribonacci $\mathcal{J}$-convergence yields

$$\mathcal{J}_{(\mathsf{P},\mathsf{Q},\mathsf{R})}\text{-}\lim_{n\to\infty} \mathbb{T}_n(x) = \ell.$$

Therefore, every Tribonacci $\mathcal{J}^*$-convergent sequence is Tribonacci $\mathcal{J}$-convergent to the same limit. □

Definition 4.2. *The ideal $\mathcal{J}$ is said to satisfy the additive property (AP) if, for every sequence (A_j) of sets belonging to $\mathcal{J}$, there exists $A \in \mathcal{J}$ such that*

$$A_j \setminus A$$

is finite for every $j \in \mathbb{N}$.

Example 4.2. *Let $\mathcal{E} = \{2n : n \in \mathbb{N}\}$ be the set of even natural numbers, and define by*

$$\mathcal{J}_{\mathcal{E}} = \{A \subseteq \mathbb{N} : A \setminus \mathcal{E} \text{ is finite}\}.$$

Thus, a set belongs to $\mathcal{J}_{\mathcal{E}}$ whenever it contains at most finitely many odd natural numbers.

Now, we first observe that $\mathcal{J}_{\mathcal{E}}$ is an ideal on $\mathbb{N}$. Clearly,

$$\varnothing \setminus \mathcal{E} = \varnothing,$$

and hence

$$\varnothing \in \mathcal{J}_{\mathcal{E}}.$$

Assume that $A \in \mathcal{J}_{\mathcal{E}}$ and $B \subseteq A$. Then,

$$B \setminus \mathcal{E} \subseteq A \setminus \mathcal{E}.$$

Since $A \setminus \mathcal{E}$ is finite, it follows that $B \setminus \mathcal{E}$ is finite. Therefore,

$$B \in \mathcal{J}_{\mathcal{E}}.$$

If $A, B \in \mathcal{J}_{\mathcal{E}}$, then

$$(A \cup B) \setminus \mathcal{E} = (A \setminus \mathcal{E}) \cup (B \setminus \mathcal{E}).$$

The right-hand side is a finite union of finite sets and is therefore finite. Hence,

$$A \cup B \in \mathcal{J}_{\mathcal{E}}.$$

Thus, $\mathcal{J}_{\mathcal{E}}$ is an ideal.

Moreover, $\mathcal{J}_{\mathcal{E}}$ is admissible because, for every $n \in \mathbb{N}$, $\{n\} \setminus \mathcal{E}$ is either empty or a singleton, and hence finite. Therefore,

$$\{n\} \in \mathcal{J}_{\mathcal{E}}.$$

The ideal is also proper. Indeed,

$$\mathbb{N} \setminus \mathcal{E} = \{2n-1 : n \in \mathbb{N}\}$$

is the infinite set of odd natural numbers. Consequently,

$$\mathbb{N} \notin \mathcal{J}_{\mathcal{E}}.$$

Now, we verify property (AP). Let (A_j) be any sequence of sets belonging to $\mathcal{J}_{\mathcal{E}}$. Then, for every $j \in \mathbb{N}$, $A_j \setminus \mathcal{E}$ is finite.

Choose $A = \mathcal{E}$. Because

$$\mathcal{E} \setminus \mathcal{E} = \varnothing,$$

then we have

$$A = \mathcal{E} \in \mathcal{J}_{\mathcal{E}}.$$

Furthermore, for every $j \in \mathbb{N}$,

$$A_j \setminus A = A_j \setminus \mathcal{E}, \text{ which is finite.}$$

Therefore, there exists

$$A \in \mathcal{J}_{\mathcal{E}}$$

such that

$$A_j \setminus A$$

is finite for every $j \in \mathbb{N}$. Hence, $\mathcal{J}_{\mathcal{E}}$ satisfies the additive property (AP).

Theorem 4.2. *Let $\mathcal{J}$ satisfy property (AP). Then Tribonacci $\mathcal{J}$-convergence implies Tribonacci $\mathcal{J}^*$-convergence.*

Proof. Assume that $x = (x_k)$ is Tribonacci $\mathcal{J}$-convergent to ℓ. Thus,

$$\mathcal{J}_{(\mathsf{P},\mathsf{Q},\mathsf{R})}\text{-}\lim_{n\to\infty} \mathbb{T}_n(x) = \ell.$$

We must prove that there exists a set

$$M = \{m_1 < m_2 < \cdots\} \in \mathcal{F}(\mathcal{J})$$

such that

$$(\mathsf{P},\mathsf{Q},\mathsf{R})\text{-}\lim_{r\to\infty} \mathbb{T}_{m_r}(x) = \ell.$$

For every integer $j \geq 2$, define

$$A_j = A_{\mathbb{T}}\left(\frac{1}{j}, \frac{1}{j}; \ell\right),$$

where

$$A_j = \left\{ n \in \mathbb{N} : \begin{array}{r} \mathsf{P}\left(\mathbb{T}_n(x) - \ell, \frac{1}{j}\right) \leq 1 - \frac{1}{j}, \\ \text{or } \mathsf{Q}\left(\mathbb{T}_n(x) - \ell, \frac{1}{j}\right) \geq \frac{1}{j}, \\ \text{or } \mathsf{R}\left(\mathbb{T}_n(x) - \ell, \frac{1}{j}\right) \geq \frac{1}{j} \end{array} \right\}.$$

Since x is Tribonacci $\mathcal{J}$-convergent to ℓ, the definition of Tribonacci $\mathcal{J}$-convergence, equivalently Theorem 3.1, gives

$$A_j \in \mathcal{J} \qquad \text{for every } j \geq 2.$$

Because $\mathcal{J}$ satisfies property (AP), there exists a set $A \in \mathcal{J}$ such that

$$A_j \setminus A$$

is finite for every $j \geq 2$.

Now, define by

$$M = \mathbb{N} \setminus A.$$

Since $A \in \mathcal{J}$, the definition of the associated filter implies that

$$M \in \mathcal{F}(\mathcal{J}).$$

We next show that M is infinite. Suppose, to the contrary, that M is finite. Since $\mathcal{J}$ is admissible, every finite subset of $\mathbb{N}$ belongs to $\mathcal{J}$. Hence,

$$M \in \mathcal{J}.$$

We already have $A \in \mathcal{J}$. Since an ideal is closed under finite unions,

$$\mathbb{N} = A \cup M \in \mathcal{J}.$$

This contradicts the assumption that $\mathcal{J}$ is proper. Therefore, M must be infinite.

Thus, M can be written as

$$M = \{m_1 < m_2 < \cdots\}.$$

Fix an arbitrary integer $j \geq 2$. Since

$$M = \mathbb{N} \setminus A,$$

we have

$$M \cap A_j = (\mathbb{N} \setminus A) \cap A_j = A_j \setminus A.$$

By property (AP), the set $A_j \setminus A$ is finite. Therefore,

$$M \cap A_j$$

is finite.

Since (m_r) is the increasing enumeration of M, there exists an integer $r_j \in \mathbb{N}$ such that

$$m_r \notin A_j \qquad \text{for every } r \geq r_j.$$

By the definition of A_j, the condition $m_r \notin A_j$ means that none of the three exceptional inequalities occurs. Consequently, for every $r \geq r_j$,

$$\mathsf{P}\left(\mathbb{T}_{m_r}(x) - \ell, \frac{1}{j}\right) > 1 - \frac{1}{j},$$

$$\mathsf{Q}\left(\mathbb{T}_{m_r}(x) - \ell, \frac{1}{j}\right) < \frac{1}{j}$$

and

$$\mathsf{R}\left(\mathbb{T}_{m_r}(x) - \ell, \frac{1}{j}\right) < \frac{1}{j}.$$

Now, we prove that

$$\mathbb{T}_{m_r}(x) \longrightarrow \ell$$

in the non-Archimedean neutrosophic norm.

Let $\varepsilon \in (0,1)$ and $t > 0$ be arbitrary. By the Archimedean property of the real numbers, there exists an integer $j \geq 2$ such that

$$\frac{1}{j} < \varepsilon \qquad \text{and} \qquad \frac{1}{j} < t.$$

Equivalently, we may choose

$$j > \max\left\{\frac{1}{\varepsilon}, \frac{1}{t}\right\}.$$

For this fixed j, there exists r_j such that, for every $r \geq r_j$,

$$\mathsf{P}\left(\mathbb{T}_{m_r}(x) - \ell, \frac{1}{j}\right) > 1 - \frac{1}{j},$$

$$\mathsf{Q}\left(\mathbb{T}_{m_r}(x)-\ell,\frac{1}{j}\right)<\frac{1}{j}$$

and

$$\mathsf{R}\left(\mathbb{T}_{m_r}(x)-\ell,\frac{1}{j}\right)<\frac{1}{j}.$$

Because the function P is nondecreasing and $\frac{1}{j}<t$, we obtain

$$\begin{aligned}\mathsf{P}\left(\mathbb{T}_{m_r}(x)-\ell,t\right)&\geq\mathsf{P}\left(\mathbb{T}_{m_r}(x)-\ell,\frac{1}{j}\right)\\&>1-\frac{1}{j}\\&>1-\varepsilon.\end{aligned}$$

The functions Q,R are nonincreasing. Therefore, since $1/j<t$, we have

$$\begin{aligned}\mathsf{Q}\left(\mathbb{T}_{m_r}(x)-\ell,t\right)&\leq\mathsf{Q}\left(\mathbb{T}_{m_r}(x)-\ell,\frac{1}{j}\right)\\&<\frac{1}{j}\\&<\varepsilon\end{aligned}$$

and

$$\begin{aligned}\mathsf{R}\left(\mathbb{T}_{m_r}(x)-\ell,t\right)&\leq\mathsf{R}\left(\mathbb{T}_{m_r}(x)-\ell,\frac{1}{j}\right)\\&<\frac{1}{j}\\&<\varepsilon.\end{aligned}$$

Hence, for every $\varepsilon\in(0,1)$ and $t>0$, there exists an integer r_j such that, for every $r\geq r_j$,

$$\mathsf{P}\left(\mathbb{T}_{m_r}(x)-\ell,t\right)>1-\varepsilon,$$

$$\mathsf{Q}\left(\mathbb{T}_{m_r}(x)-\ell,t\right)<\varepsilon$$

and

$$\mathsf{R}\left(\mathbb{T}_{m_r}(x)-\ell,t\right)<\varepsilon.$$

Therefore,

$$(\mathsf{P},\mathsf{Q},\mathsf{R})\text{-}\lim_{r\to\infty}\mathbb{T}_{m_r}(x)=\ell.$$

Since

$$M=\{m_1<m_2<\cdots\}\in\mathcal{F}(\mathcal{J}),$$

the definition of Tribonacci $\mathcal{J}^*$-convergence yields

$$\mathcal{J}^*_{(\mathsf{P},\mathsf{Q},\mathsf{R})}\text{-}\lim_{n\to\infty}\mathbb{T}_n(x)=\ell.$$

Thus, Tribonacci $\mathcal{J}$-convergence implies Tribonacci $\mathcal{J}^*$-convergence whenever $\mathcal{J}$ satisfies property (AP). □

Corollary 4.1. *Let $\mathcal{J}$ be a proper admissible ideal of $\mathbb{N}$ satisfying property (AP). Then a sequence $x=(x_k)$ is Tribonacci $\mathcal{J}$-convergent to ℓ if and only if it is Tribonacci $\mathcal{J}^*$-convergent to ℓ.*

Proof. Suppose first that x is Tribonacci $\mathcal{J}$-convergent to ℓ. By Theorem 4.2, x is Tribonacci $\mathcal{J}^*$-convergent to ℓ. Conversely, suppose that x is Tribonacci $\mathcal{J}^*$-convergent to ℓ. By Theorem 4.1, x is Tribonacci $\mathcal{J}$-convergent to ℓ. □

Remark 4.1. *The additive property (AP) is required only for the implication from Tribonacci $\mathcal{J}$-convergence to Tribonacci $\mathcal{J}^*$-convergence. The reverse implication holds for every proper admissible ideal by Theorem 4.1.*

5. Tribonacci Ideal-Cauchy Sequences and Completeness

In this section, we introduce the notion of a Tribonacci $\mathcal{J}$-Cauchy sequence in a non-Archimedean neutrosophic normed space. We first prove that every Tribonacci $\mathcal{J}$-convergent sequence is Tribonacci $\mathcal{J}$-Cauchy. We then establish the converse under the assumptions that the underlying space is complete and the ideal $\mathcal{J}$ satisfies property (AP).

Definition 5.1. *A sequence $x = (x_k)$ is said to be Tribonacci $\mathcal{J}$-Cauchy if, for every $\varepsilon \in (0,1)$ and $t > 0$, there exists $p \in \mathbb{N}$ such that*

$$C_{\mathbb{T}}(\varepsilon, t; p) = \left\{ n \in \mathbb{N} : \begin{array}{c} \mathsf{P}(\mathbb{T}_n(x) - \mathbb{T}_p(x), t) \leq 1 - \varepsilon \\ \text{or } \mathsf{Q}(\mathbb{T}_n(x) - \mathbb{T}_p(x), t) \geq \varepsilon \\ \text{or } \mathsf{R}(\mathbb{T}_n(x) - \mathbb{T}_p(x), t) \geq \varepsilon \end{array} \right\} \in \mathcal{J}.$$

Example 5.1. *Let $(X, \mathsf{P}, \mathsf{Q}, \mathsf{R}, *, \diamond)$ be a non-Archimedean neutrosophic normed space, and let $u \in X$ be a fixed nonzero vector. Consider the ideal*

$$\mathcal{J}_d = \{A \subseteq \mathbb{N} : d(A) = 0\},$$

where $d(A)$ denotes the natural density of A.

Let

$$S = \{2^m : m \in \mathbb{N}\}.$$

Define a sequence $z = (z_n)$ in X by

$$z_n = \begin{cases} u, & n \in S, \\ 0, & n \notin S. \end{cases}$$

Suppose that the Tribonacci matrix $\mathbb{T} = (a_{nk})$ is lower triangular and satisfies

$$a_{nn} \neq 0 \text{ for every } n \in \mathbb{N}.$$

Then, there exists a unique sequence $x = (x_k)$ in X such that

$$\mathbb{T}_n(x) = z_n \text{ for every } n \in \mathbb{N}.$$

Indeed, the sequence x may be defined recursively as

$$x_1 = a_{11}^{-1} z_1$$

and

$$x_n = a_{nn}^{-1} \left(z_n - \sum_{k=1}^{n-1} a_{nk} x_k \right), \qquad n \geq 2.$$

Now, we prove that x is Tribonacci $\mathcal{J}_d$-Cauchy. Let $\varepsilon \in (0,1)$ and $t > 0$ be arbitrary. Choose an index $p \notin S$; for example, take $p = 3$. Then,

$$\mathbb{T}_p(x) = z_p = 0.$$

If $n \notin S$, then

$$\mathbb{T}_n(x) = z_n = 0,$$

and hence

$$\mathbb{T}_n(x) - \mathbb{T}_p(x) = 0.$$

Therefore,

$$\mathsf{P}\big(\mathbb{T}_n(x) - \mathbb{T}_p(x), t\big) = 1,$$
$$\mathsf{Q}\big(\mathbb{T}_n(x) - \mathbb{T}_p(x), t\big) = 0$$

and

$$\mathsf{R}\big(\mathbb{T}_n(x) - \mathbb{T}_p(x), t\big) = 0.$$

Thus, no index outside S belongs to the exceptional Cauchy set. Consequently,

$$C_{\mathbb{T}}(\varepsilon, t; p) \subseteq S.$$

The set S has natural density zero. Indeed,

$$|S \cap \{1, 2, \ldots, n\}| \leq \lfloor \log_2 n \rfloor + 1,$$

and hence

$$\frac{|S \cap \{1, 2, \ldots, n\}|}{n} \leq \frac{\lfloor \log_2 n \rfloor + 1}{n} \longrightarrow 0.$$

Therefore,

$$S \in \mathcal{J}_d.$$

Since $\mathcal{J}_d$ is hereditary, it follows that

$$C_{\mathbb{T}}(\varepsilon, t; p) \in \mathcal{J}_d.$$

Since $\varepsilon \in (0, 1)$ and $t > 0$ were arbitrary, x is Tribonacci $\mathcal{J}_d$-Cauchy.

However, the transformed sequence $\big(\mathbb{T}_n(x)\big)$ is not an ordinary Cauchy sequence. Indeed, for every $m \in \mathbb{N}$, choose

$$r_m = 2^m \qquad \text{and} \qquad s_m = 2^m + 1.$$

Then,

$$\mathbb{T}_{r_m}(x) = u$$

and

$$\mathbb{T}_{s_m}(x) = 0,$$

so that

$$\mathbb{T}_{r_m}(x) - \mathbb{T}_{s_m}(x) = u.$$

Because $u \neq 0$, there exist $\varepsilon_0 \in (0, 1)$ and $t_0 > 0$ such that at least one of the inequalities

$$\mathsf{P}(u, t_0) \leq 1 - \varepsilon_0,$$

$$\mathsf{Q}(u, t_0) \geq \varepsilon_0$$

or

$$\mathsf{R}(u, t_0) \geq \varepsilon_0$$

holds. Since $r_m, s_m \to \infty$, the ordinary Cauchy condition fails for arbitrarily large indices.

Therefore, x is Tribonacci $\mathcal{J}_d$-Cauchy, although it is not an ordinary Tribonacci Cauchy sequence.

Theorem 5.1. *Let $(X, \mathsf{P}, \mathsf{Q}, \mathsf{R}, *, \diamond)$ be a non-Archimedean neutrosophic normed space, and let $\mathcal{J}$ be a proper admissible ideal of $\mathbb{N}$. Every Tribonacci $\mathcal{J}$-convergent sequence in X is Tribonacci $\mathcal{J}$-Cauchy.*

Proof. Let $x = (x_k)$ be a Tribonacci $\mathcal{J}$-convergent sequence. Suppose that

$$\mathcal{J}_{(\mathsf{P},\mathsf{Q},\mathsf{R})}\text{-}\lim_{n\to\infty} \mathbb{T}_n(x) = \ell$$

for some $\ell \in X$.

We prove that x is Tribonacci $\mathcal{J}$-Cauchy.

Suppose that x is Tribonacci $\mathcal{J}$-convergent to ℓ. Let $\varepsilon \in (0, 1)$ and $t > 0$. Choose $\delta \in (0, 1)$ such that

$$(1 - \delta) * (1 - \delta) > 1 - \varepsilon$$

and

$$\delta \diamond \delta < \varepsilon.$$

Define the good-index set

$$G = \left\{ n \in \mathbb{N} : \begin{array}{l} \mathsf{P}(\mathbb{T}_n(x) - \ell, t) > 1 - \delta, \\ \mathsf{Q}(\mathbb{T}_n(x) - \ell, t) < \delta, \\ \mathsf{R}(\mathbb{T}_n(x) - \ell, t) < \delta \end{array} \right\}.$$

Since x is Tribonacci $\mathcal{J}$-convergent to ℓ, Theorem 3.1 implies that

$$G \in \mathcal{F}(\mathcal{J}).$$

Because $\mathcal{F}(\mathcal{J})$ is a filter, the empty set does not belong to $\mathcal{F}(\mathcal{J})$. Hence,

$$G \neq \varnothing.$$

Choose and fix an index

$$p \in G.$$

By the definition of G, we have

$$\begin{aligned} \mathsf{P}(\mathbb{T}_p(x) - \ell, t) &> 1 - \delta, \\ \mathsf{Q}(\mathbb{T}_p(x) - \ell, t) &< \delta \end{aligned}$$

and

$$\mathsf{R}(\mathbb{T}_p(x) - \ell, t) < \delta.$$

By Theorem 3.2, we have

$$\begin{aligned} \mathsf{P}(\ell - \mathbb{T}_p(x), t) &> 1 - \delta, \\ \mathsf{Q}(\ell - \mathbb{T}_p(x), t) &< \delta \end{aligned}$$

and

$$\mathsf{R}(\ell - \mathbb{T}_p(x), t) < \delta.$$

Now, let $n \in G$. Then

$$\begin{aligned} \mathsf{P}(\mathbb{T}_n(x) - \ell, t) &> 1 - \delta, \\ \mathsf{Q}(\mathbb{T}_n(x) - \ell, t) &< \delta \end{aligned}$$

and

$$\mathsf{R}(\mathbb{T}_n(x) - \ell, t) < \delta.$$

We obtain

$$\begin{aligned} &\mathsf{P}\big(\mathbb{T}_n(x) - \mathbb{T}_p(x), t\big) \\ &\quad = \mathsf{P}\left(\big(\mathbb{T}_n(x) - \ell\big) + \big(\ell - \mathbb{T}_p(x)\big), \max\{t, t\}\right) \\ &\quad \geq \mathsf{P}\big(\mathbb{T}_n(x) - \ell, t\big) * \mathsf{P}\big(\ell - \mathbb{T}_p(x), t\big) \\ &\quad > (1 - \delta) * (1 - \delta) \\ &\quad > 1 - \varepsilon. \end{aligned}$$

Similarly,

$$\begin{aligned} &\mathsf{Q}\big(\mathbb{T}_n(x) - \mathbb{T}_p(x), t\big) \\ &\quad \leq \mathsf{Q}\big(\mathbb{T}_n(x) - \ell, t\big) \diamond \mathsf{Q}\big(\ell - \mathbb{T}_p(x), t\big) \\ &\quad < \delta \diamond \delta \\ &\quad < \varepsilon. \end{aligned}$$

In the same way,

$$\begin{aligned} &\mathsf{R}\big(\mathbb{T}_n(x) - \mathbb{T}_p(x), t\big) \\ &\quad \leq \mathsf{R}\big(\mathbb{T}_n(x) - \ell, t\big) \diamond \mathsf{R}\big(\ell - \mathbb{T}_p(x), t\big) \\ &\quad < \delta \diamond \delta \\ &\quad < \varepsilon. \end{aligned}$$

Thus, every $n \in G$ satisfies

$$\mathsf{P}\big(\mathbb{T}_n(x) - \mathbb{T}_p(x), t\big) > 1 - \varepsilon,$$

$$\mathsf{Q}\big(\mathbb{T}_n(x) - \mathbb{T}_p(x), t\big) < \varepsilon$$

and

$$\mathsf{R}\big(\mathbb{T}_n(x) - \mathbb{T}_p(x), t\big) < \varepsilon.$$

Now, consider the exceptional Cauchy set

$$C_{\mathbb{T}}(\varepsilon, t; p) = \left\{ n \in \mathbb{N} : \begin{array}{c} \mathsf{P}\big(\mathbb{T}_n(x) - \mathbb{T}_p(x), t\big) \leq 1 - \varepsilon, \\ \text{or } \mathsf{Q}\big(\mathbb{T}_n(x) - \mathbb{T}_p(x), t\big) \geq \varepsilon, \\ \text{or } \mathsf{R}\big(\mathbb{T}_n(x) - \mathbb{T}_p(x), t\big) \geq \varepsilon \end{array} \right\}.$$

Since every index in G satisfies all three desired Cauchy inequalities, no index in G belongs to $C_{\mathbb{T}}(\varepsilon, t; p)$. Therefore,

$$G \cap C_{\mathbb{T}}(\varepsilon, t; p) = \varnothing,$$

which is equivalent to

$$C_{\mathbb{T}}(\varepsilon, t; p) \subseteq \mathbb{N} \setminus G.$$

Since

$$G \in \mathcal{F}(\mathcal{J}),$$

the definition of the associated filter gives

$$\mathbb{N} \setminus G \in \mathcal{J}.$$

Because $\mathcal{J}$ is hereditary and

$$C_{\mathbb{T}}(\varepsilon, t; p) \subseteq \mathbb{N} \setminus G,$$

it follows that

$$C_{\mathbb{T}}(\varepsilon, t; p) \in \mathcal{J}.$$

Thus, for every $\varepsilon \in (0, 1)$ and $t > 0$, there exists an index $p \in \mathbb{N}$ such that

$$C_{\mathbb{T}}(\varepsilon, t; p) \in \mathcal{J}.$$

Therefore, x is Tribonacci $\mathcal{J}$-Cauchy. □

We now establish the converse under an additional assumption on the ideal.

Theorem 5.2. *Let $(X, \mathsf{P}, \mathsf{Q}, \mathsf{R}, *, \diamond)$ be complete, and suppose that $\mathcal{J}$ satisfies property (AP). Then a sequence is Tribonacci $\mathcal{J}$-convergent if and only if it is Tribonacci $\mathcal{J}$-Cauchy.*

Proof. We prove the two implications separately.

Necessity. Suppose that $x = (x_k)$ is Tribonacci $\mathcal{J}$-convergent. Then, by Theorem 5.1, every Tribonacci $\mathcal{J}$-convergent sequence is Tribonacci $\mathcal{J}$-Cauchy. Hence, x is Tribonacci $\mathcal{J}$-Cauchy.

Sufficiency. Conversely, suppose that $x = (x_k)$ is Tribonacci $\mathcal{J}$-Cauchy. We prove that x is Tribonacci $\mathcal{J}$-convergent.

For each $j \geq 2$, there exists $\eta_j \in (0, 1)$ such that

$$(1 - \eta_j) * (1 - \eta_j) > 1 - \frac{1}{j}$$

and

$$\eta_j \diamond \eta_j < \frac{1}{j}.$$

Since x is Tribonacci $\mathcal{J}$-Cauchy, for every $j \geq 2$ there exists an index $p_j \in \mathbb{N}$ such that the exceptional set

$$A_j = C_{\mathbb{T}}\left(\eta_j, \frac{1}{j}; p_j\right) \in \mathcal{J}.$$

More explicitly,

$$A_j = \left\{ n \in \mathbb{N} : \begin{array}{r} \mathsf{P}\left(\mathbb{T}_n(x) - \mathbb{T}_{p_j}(x), \frac{1}{j}\right) \le 1 - \eta_j, \\ \text{or } \mathsf{Q}\left(\mathbb{T}_n(x) - \mathbb{T}_{p_j}(x), \frac{1}{j}\right) \ge \eta_j, \\ \text{or } \mathsf{R}\left(\mathbb{T}_n(x) - \mathbb{T}_{p_j}(x), \frac{1}{j}\right) \ge \eta_j \end{array} \right\}.$$

Thus,

$$A_j \in \mathcal{J} \qquad \text{for every } j \ge 2.$$

Because $\mathcal{J}$ satisfies property (AP), there exists a set $A \in \mathcal{J}$ such that

$$A_j \setminus A$$

is finite for every $j \ge 2$.

Define

$$M = \mathbb{N} \setminus A.$$

Since $A \in \mathcal{J}$, the definition of the associated filter implies that

$$M \in \mathcal{F}(\mathcal{J}).$$

We now show that M is infinite. Suppose, to the contrary, that M is finite. Since $\mathcal{J}$ is admissible, every finite subset of $\mathbb{N}$ belongs to $\mathcal{J}$. Therefore,

$$M \in \mathcal{J}.$$

Since $A \in \mathcal{J}$ and an ideal is closed under finite unions, we would have

$$\mathbb{N} = A \cup M \in \mathcal{J},$$

which contradicts the fact that $\mathcal{J}$ is proper. Hence, M is infinite.

We may therefore write

$$M = \{m_1 < m_2 < m_3 < \cdots\}.$$

We shall prove that the sequence of Tribonacci transforms

$$(\mathbb{T}_{m_r}(x))$$

is an ordinary Cauchy sequence in the NA-NNS.

Fix an integer $j \ge 2$. Since

$$M = \mathbb{N} \setminus A,$$

we have

$$\begin{aligned} M \cap A_j &= (\mathbb{N} \setminus A) \cap A_j \\ &= A_j \setminus A. \end{aligned}$$

By the choice of A, the set $A_j \setminus A$ is finite. Hence,

$$M \cap A_j$$

is finite.

Since (m_r) is the increasing enumeration of M, there exists an integer $r_j \in \mathbb{N}$ such that

$$m_r \notin A_j \qquad \text{for every } r \ge r_j.$$

Consequently, whenever $r, s \ge r_j$, both

$$m_r \notin A_j \qquad \text{and} \qquad m_s \notin A_j.$$

By the definition of A_j, the relation $m_r \notin A_j$ means that none of the three exceptional inequalities occurs. Therefore,

$$\mathsf{P}\left(\mathbb{T}_{m_r}(x) - \mathbb{T}_{p_j}(x), \frac{1}{j}\right) > 1 - \eta_j,$$

$$\mathsf{Q}\left(\mathbb{T}_{m_r}(x)-\mathbb{T}_{p_j}(x),\frac{1}{j}\right)<\eta_j$$

and

$$\mathsf{R}\left(\mathbb{T}_{m_r}(x)-\mathbb{T}_{p_j}(x),\frac{1}{j}\right)<\eta_j.$$

Similarly,

$$\mathsf{P}\left(\mathbb{T}_{m_s}(x)-\mathbb{T}_{p_j}(x),\frac{1}{j}\right)>1-\eta_j,$$

$$\mathsf{Q}\left(\mathbb{T}_{m_s}(x)-\mathbb{T}_{p_j}(x),\frac{1}{j}\right)<\eta_j$$

and

$$\mathsf{R}\left(\mathbb{T}_{m_s}(x)-\mathbb{T}_{p_j}(x),\frac{1}{j}\right)<\eta_j.$$

We need the corresponding estimates for

$$\mathbb{T}_{p_j}(x)-\mathbb{T}_{m_s}(x).$$

From Theorem 3.2, we have

$$\mathsf{P}\left(\mathbb{T}_{p_j}(x)-\mathbb{T}_{m_s}(x),\frac{1}{j}\right)>1-\eta_j,$$

$$\mathsf{Q}\left(\mathbb{T}_{p_j}(x)-\mathbb{T}_{m_s}(x),\frac{1}{j}\right)<\eta_j$$

and

$$\mathsf{R}\left(\mathbb{T}_{p_j}(x)-\mathbb{T}_{m_s}(x),\frac{1}{j}\right)<\eta_j.$$

Now observe that

$$\begin{aligned}\mathbb{T}_{m_r}(x)-\mathbb{T}_{m_s}(x)&=\big(\mathbb{T}_{m_r}(x)-\mathbb{T}_{p_j}(x)\big)\\&\quad+\big(\mathbb{T}_{p_j}(x)-\mathbb{T}_{m_s}(x)\big).\end{aligned}$$

Using the non-Archimedean triangle property for the truth-membership function and

$$\max\left\{\frac{1}{j},\frac{1}{j}\right\}=\frac{1}{j},$$

we obtain

$$\begin{aligned}&\mathsf{P}\left(\mathbb{T}_{m_r}(x)-\mathbb{T}_{m_s}(x),\frac{1}{j}\right)\\&\geq\mathsf{P}\left(\mathbb{T}_{m_r}(x)-\mathbb{T}_{p_j}(x),\frac{1}{j}\right)*\mathsf{P}\left(\mathbb{T}_{p_j}(x)-\mathbb{T}_{m_s}(x),\frac{1}{j}\right)\\&>(1-\eta_j)*(1-\eta_j)\\&>1-\frac{1}{j}.\end{aligned}$$

Similarly,

$$\begin{aligned}&\mathsf{Q}\left(\mathbb{T}_{m_r}(x)-\mathbb{T}_{m_s}(x),\frac{1}{j}\right)\\&\leq\mathsf{Q}\left(\mathbb{T}_{m_r}(x)-\mathbb{T}_{p_j}(x),\frac{1}{j}\right)\diamond\mathsf{Q}\left(\mathbb{T}_{p_j}(x)-\mathbb{T}_{m_s}(x),\frac{1}{j}\right)\\&<\eta_j\diamond\eta_j\\&<\frac{1}{j},\end{aligned}$$

and

$$\begin{aligned}
&\mathsf{R}\left(\mathbb{T}_{m_r}(x)-\mathbb{T}_{m_s}(x),\frac{1}{j}\right)\\
&\quad\le \mathsf{R}\left(\mathbb{T}_{m_r}(x)-\mathbb{T}_{p_j}(x),\frac{1}{j}\right)\diamond \mathsf{R}\left(\mathbb{T}_{p_j}(x)-\mathbb{T}_{m_s}(x),\frac{1}{j}\right)\\
&\quad< \eta_j\diamond\eta_j\\
&\quad<\frac{1}{j}.
\end{aligned}$$

Therefore, for every fixed $j\ge 2$, there exists r_j such that, for all $r,s\ge r_j$,

$$\mathsf{P}\left(\mathbb{T}_{m_r}(x)-\mathbb{T}_{m_s}(x),\frac{1}{j}\right)>1-\frac{1}{j},$$

$$\mathsf{Q}\left(\mathbb{T}_{m_r}(x)-\mathbb{T}_{m_s}(x),\frac{1}{j}\right)<\frac{1}{j}$$

and

$$\mathsf{R}\left(\mathbb{T}_{m_r}(x)-\mathbb{T}_{m_s}(x),\frac{1}{j}\right)<\frac{1}{j}.$$

Now, we verify the ordinary Cauchy condition. Let $\varepsilon\in(0,1)$ and $t>0$ be arbitrary. Choose an integer $j\ge 2$ sufficiently large that

$$\frac{1}{j}<\varepsilon \qquad \text{and} \qquad \frac{1}{j}<t.$$

For example, it is enough to choose

$$j>\max\left\{\frac{1}{\varepsilon},\frac{1}{t}\right\}.$$

For all $r,s\ge r_j$, the preceding inequalities hold. Since P is nondecreasing and $1/j<t$, we obtain

$$\begin{aligned}
\mathsf{P}\left(\mathbb{T}_{m_r}(x)-\mathbb{T}_{m_s}(x),t\right)&\ge \mathsf{P}\left(\mathbb{T}_{m_r}(x)-\mathbb{T}_{m_s}(x),\frac{1}{j}\right)\\
&>1-\frac{1}{j}\\
&>1-\varepsilon.
\end{aligned}$$

Since $\mathsf{Q}\ \mathsf{R}$ is nonincreasing, we have

$$\begin{aligned}
\mathsf{Q}\left(\mathbb{T}_{m_r}(x)-\mathbb{T}_{m_s}(x),t\right)&\le \mathsf{Q}\left(\mathbb{T}_{m_r}(x)-\mathbb{T}_{m_s}(x),\frac{1}{j}\right)\\
&<\frac{1}{j}\\
&<\varepsilon,
\end{aligned}$$

and

$$\begin{aligned}
\mathsf{R}\left(\mathbb{T}_{m_r}(x)-\mathbb{T}_{m_s}(x),t\right)&\le \mathsf{R}\left(\mathbb{T}_{m_r}(x)-\mathbb{T}_{m_s}(x),\frac{1}{j}\right)\\
&<\frac{1}{j}\\
&<\varepsilon.
\end{aligned}$$

Thus, for every $\varepsilon\in(0,1)$ and $t>0$, there exists $r_j\in\mathbb{N}$ such that, for all $r,s\ge r_j$,

$$\mathsf{P}\left(\mathbb{T}_{m_r}(x)-\mathbb{T}_{m_s}(x),t\right)>1-\varepsilon,$$

$$\mathsf{Q}\left(\mathbb{T}_{m_r}(x)-\mathbb{T}_{m_s}(x),t\right)<\varepsilon$$

and

$$\mathsf{R}\left(\mathbb{T}_{m_r}(x) - \mathbb{T}_{m_s}(x), t\right) < \varepsilon.$$

Therefore,

$$(\mathbb{T}_{m_r}(x))$$

is an ordinary Cauchy sequence in X.

Since X is complete, there exists $\ell \in X$ such that

$$(\mathsf{P}, \mathsf{Q}, \mathsf{R})\text{-}\lim_{r\to\infty} \mathbb{T}_{m_r}(x) = \ell.$$

Moreover,

$$M = \{m_1 < m_2 < \cdots\} \in \mathcal{F}(\mathcal{J}).$$

Hence, by the definition of Tribonacci $\mathcal{J}^*$-convergence,

$$\mathcal{J}^*_{(\mathsf{P},\mathsf{Q},\mathsf{R})}\text{-}\lim_{n\to\infty} \mathbb{T}_n(x) = \ell.$$

Finally, by Theorem 4.1, every Tribonacci $\mathcal{J}^*$-convergent sequence is Tribonacci $\mathcal{J}$-convergent to the same limit. Therefore,

$$\mathcal{J}_{(\mathsf{P},\mathsf{Q},\mathsf{R})}\text{-}\lim_{n\to\infty} \mathbb{T}_n(x) = \ell.$$

Thus, x is Tribonacci $\mathcal{J}$-convergent.

Hence, a sequence is Tribonacci $\mathcal{J}$-convergent if and only if it is Tribonacci $\mathcal{J}$-Cauchy. □

6. Preservation Properties of Tribonacci Ideal Convergence

In this section, we investigate two fundamental preservation properties of Tribonacci ideal convergence in non-Archimedean neutrosophic normed spaces. First, we show that Tribonacci ideal convergence is preserved by neutrosophically continuous linear operators. Second, we establish a monotonicity property with respect to the inclusion of admissible ideals.

6.1. **Preservation under Continuous Linear Operators.** We begin by introducing the appropriate notion of continuity for linear operators between non-Archimedean neutrosophic normed spaces.

Definition 6.1. *Let X and Y be NA-NNSs. A linear operator $L : X \to Y$ is called neutrosophically continuous at zero if, for every $\varepsilon \in (0,1)$ and $t > 0$, there exist $\delta \in (0,1)$ and $s > 0$ such that*

$$\mathsf{P}_X(u, s) > 1 - \delta, \quad \mathsf{Q}_X(u, s) < \delta, \quad \mathsf{R}_X(u, s) < \delta$$

imply

$$\mathsf{P}_Y(Lu, t) > 1 - \varepsilon, \quad \mathsf{Q}_Y(Lu, t) < \varepsilon, \quad \mathsf{R}_Y(Lu, t) < \varepsilon.$$

Example 6.1. *Let $(X, \|\cdot\|)$ be a non-Archimedean normed space over a non-Archimedean valued field $(\mathbb{K}, |\cdot|)$. Define*

$$\mathsf{P}(x, t) = \frac{t}{t + \|x\|}, \qquad \mathsf{Q}(x, t) = \frac{\|x\|}{t + \|x\|}$$

and

$$\mathsf{R}(x, t) = \frac{\|x\|}{t + \|x\|}, \qquad x \in X, \quad t > 0.$$

Let

$$a * b = \min\{a, b\}, \qquad a \diamond b = \max\{a, b\}.$$

Then, $(X, \mathsf{P}, \mathsf{Q}, \mathsf{R}, *, \diamond)$ is a non-Archimedean neutrosophic normed space.

Fix $c \in \mathbb{K}$ and define

$$L : X \longrightarrow X$$

by

$$Lx = cx, \qquad x \in X.$$

Clearly, L is a linear operator.
Now, we will show that L is neutrosophically continuous at zero. If $c = 0$, then

$$Lu = 0$$

for every $u \in X$, and hence

$$\mathsf{P}(Lu, t) = 1, \qquad \mathsf{Q}(Lu, t) = 0, \qquad \mathsf{R}(Lu, t) = 0.$$

Thus, the continuity condition is immediately satisfied.
Now, suppose that $c \neq 0$. Let $\varepsilon \in (0, 1)$ and $t > 0$ be arbitrary. Choose

$$\delta = \varepsilon \qquad \text{and} \qquad s = \frac{t}{|c|}.$$

Assume that, for some $u \in X$,

$$\mathsf{P}(u, s) > 1 - \delta, \qquad \mathsf{Q}(u, s) < \delta, \qquad \mathsf{R}(u, s) < \delta.$$

After applying the homogeneity of the non-Archimedean norm, we obtain

$$\|Lu\| = \|cu\| = |c|\|u\|.$$

Therefore,

$$\begin{aligned}
\mathsf{P}(Lu, t) &= \frac{t}{t + \|cu\|} \\
&= \frac{t}{t + |c|\|u\|} \\
&= \frac{t/|c|}{t/|c| + \|u\|} \\
&= \mathsf{P}\left(u, \frac{t}{|c|}\right) \\
&= \mathsf{P}(u, s) \\
&> 1 - \delta \\
&= 1 - \varepsilon.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathsf{Q}(Lu, t) &= \frac{\|cu\|}{t + \|cu\|} \\
&= \frac{|c|\|u\|}{t + |c|\|u\|} \\
&= \frac{\|u\|}{t/|c| + \|u\|} \\
&= \mathsf{Q}\left(u, \frac{t}{|c|}\right) \\
&= \mathsf{Q}(u, s) \\
&< \delta \\
&= \varepsilon,
\end{aligned}$$

and

$$\begin{aligned}\mathsf{R}(Lu,t) &= \frac{\|cu\|}{t+\|cu\|}\\ &= \frac{|c|\|u\|}{t+|c|\|u\|}\\ &= \frac{\|u\|}{t/|c|+\|u\|}\\ &= \mathsf{R}\left(u,\frac{t}{|c|}\right)\\ &= \mathsf{R}(u,s)\\ &< \delta\\ &= \varepsilon.\end{aligned}$$

Hence,

$$\mathsf{P}(u,s) > 1-\delta, \qquad \mathsf{Q}(u,s) < \delta, \qquad \mathsf{R}(u,s) < \delta$$

imply

$$\mathsf{P}(Lu,t) > 1-\varepsilon, \qquad \mathsf{Q}(Lu,t) < \varepsilon, \qquad \mathsf{R}(Lu,t) < \varepsilon.$$

Therefore, $Lx = cx$ is neutrosophically continuous at zero.

Theorem 6.1. *Let $(X, \mathsf{P}_X, \mathsf{Q}_X, \mathsf{R}_X, *_X, \diamond_X)$ and $(Y, \mathsf{P}_Y, \mathsf{Q}_Y, \mathsf{R}_Y, *_Y, \diamond_Y)$ be non-Archimedean neutrosophic normed spaces over the same non-Archimedean valued field $\mathbb{K}$. Let $\mathcal{J}$ be a proper admissible ideal of $\mathbb{N}$, and let $L : X \to Y$ be a neutrosophically continuous linear operator.*

For a sequence $x = (x_k)$ in X, define

$$Lx = (Lx_k).$$

If

$$\mathcal{J}_{(\mathsf{P}_X,\mathsf{Q}_X,\mathsf{R}_X)}\text{-}\lim_{n\to\infty} \mathbb{T}_n(x) = \ell,$$

then

$$\mathcal{J}_{(\mathsf{P}_Y,\mathsf{Q}_Y,\mathsf{R}_Y)}\text{-}\lim_{n\to\infty} \mathbb{T}_n(Lx) = L\ell.$$

Proof. Assume that

$$\mathcal{J}_{(\mathsf{P}_X,\mathsf{Q}_X,\mathsf{R}_X)}\text{-}\lim_{n\to\infty} \mathbb{T}_n(x) = \ell.$$

We prove that the Tribonacci transform of the sequence

$$Lx = (Lx_k)$$

is $\mathcal{J}$-convergent to $L\ell$ in Y.

Let $\varepsilon \in (0,1)$ and $t > 0$ be arbitrary. Since L is neutrosophically continuous at zero, there exist $\delta \in (0,1)$ and $s > 0$ such that, for every $u \in X$,

$$\mathsf{P}_X(u,s) > 1-\delta, \qquad \mathsf{Q}_X(u,s) < \delta$$

and

$$\mathsf{R}_X(u,s) < \delta$$

imply

$$\mathsf{P}_Y(Lu,t) > 1-\varepsilon, \qquad \mathsf{Q}_Y(Lu,t) < \varepsilon$$

and

$$\mathsf{R}_Y(Lu,t) < \varepsilon.$$

First, we verify that the linear operator L commutes with the Tribonacci transformation. For every $n \in \mathbb{N}$, we have

$$\begin{aligned}\mathbb{T}_n(Lx) &= \sum_{k=1}^{n} a_{nk} L x_k \\ &= L\left(\sum_{k=1}^{n} a_{nk} x_k\right) \\ &= L\big(\mathbb{T}_n(x)\big),\end{aligned}$$

where the second equality follows from the linearity of L. Therefore,

$$\begin{aligned}\mathbb{T}_n(Lx) - L\ell &= L\big(\mathbb{T}_n(x)\big) - L\ell \\ &= L\big(\mathbb{T}_n(x) - \ell\big).\end{aligned}$$

Define the good-index set in X by

$$G_X = \left\{ n \in \mathbb{N} : \begin{array}{c} \mathsf{P}_X(\mathbb{T}_n(x) - \ell, s) > 1 - \delta, \\ \mathsf{Q}_X(\mathbb{T}_n(x) - \ell, s) < \delta, \\ \mathsf{R}_X(\mathbb{T}_n(x) - \ell, s) < \delta \end{array} \right\}.$$

Since

$$\mathcal{J}_{(\mathsf{P}_X, \mathsf{Q}_X, \mathsf{R}_X)}\text{-}\lim_{n\to\infty} \mathbb{T}_n(x) = \ell,$$

Theorem 3.1 implies that

$$G_X \in \mathcal{F}(\mathcal{J}).$$

Now, let $n \in G_X$. Then

$$\begin{gathered}\mathsf{P}_X(\mathbb{T}_n(x) - \ell, s) > 1 - \delta, \\ \mathsf{Q}_X(\mathbb{T}_n(x) - \ell, s) < \delta\end{gathered}$$

and

$$\mathsf{R}_X(\mathbb{T}_n(x) - \ell, s) < \delta.$$

Using the neutrosophic continuity of L to

$$u = \mathbb{T}_n(x) - \ell.$$

We obtain

$$\begin{gathered}\mathsf{P}_Y\left(L\big(\mathbb{T}_n(x) - \ell\big), t\right) > 1 - \varepsilon, \\ \mathsf{Q}_Y\left(L\big(\mathbb{T}_n(x) - \ell\big), t\right) < \varepsilon\end{gathered}$$

and

$$\mathsf{R}_Y\left(L\big(\mathbb{T}_n(x) - \ell\big), t\right) < \varepsilon.$$

Since

$$L\big(\mathbb{T}_n(x) - \ell\big) = \mathbb{T}_n(Lx) - L\ell,$$

it follows that

$$\begin{gathered}\mathsf{P}_Y\big(\mathbb{T}_n(Lx) - L\ell, t\big) > 1 - \varepsilon, \\ \mathsf{Q}_Y\big(\mathbb{T}_n(Lx) - L\ell, t\big) < \varepsilon\end{gathered}$$

and

$$\mathsf{R}_Y\big(\mathbb{T}_n(Lx) - L\ell, t\big) < \varepsilon.$$

Now, we define the corresponding good-index set in Y by

$$G_Y = \left\{ n \in \mathbb{N} : \begin{array}{c} \mathsf{P}_Y(\mathbb{T}_n(Lx) - L\ell, t) > 1 - \varepsilon, \\ \mathsf{Q}_Y(\mathbb{T}_n(Lx) - L\ell, t) < \varepsilon, \\ \mathsf{R}_Y(\mathbb{T}_n(Lx) - L\ell, t) < \varepsilon \end{array} \right\}.$$

The preceding argument shows that

$$G_X \subseteq G_Y.$$

Since

$$G_X \in \mathcal{F}(\mathcal{J})$$

and a filter is upward closed, we conclude that

$$G_Y \in \mathcal{F}(\mathcal{J}).$$

Equivalently, the exceptional set

$$A_Y(\varepsilon, t; L\ell) = \mathbb{N} \setminus G_Y \in \mathcal{J}.$$

Explicitly,

$$A_Y(\varepsilon, t; L\ell) = \left\{ n \in \mathbb{N} : \begin{array}{l} \mathsf{P}_Y(\mathbb{T}_n(Lx) - L\ell, t) \leq 1 - \varepsilon, \\ \text{or } \mathsf{Q}_Y(\mathbb{T}_n(Lx) - L\ell, t) \geq \varepsilon, \\ \text{or } \mathsf{R}_Y(\mathbb{T}_n(Lx) - L\ell, t) \geq \varepsilon \end{array} \right\}.$$

Alternatively, this can be seen directly from the inclusion

$$A_Y(\varepsilon, t; L\ell) \subseteq A_X(\delta, s; \ell),$$

where

$$A_X(\delta, s; \ell) = \left\{ n \in \mathbb{N} : \begin{array}{l} \mathsf{P}_X(\mathbb{T}_n(x) - \ell, s) \leq 1 - \delta, \\ \text{or } \mathsf{Q}_X(\mathbb{T}_n(x) - \ell, s) \geq \delta, \\ \text{or } \mathsf{R}_X(\mathbb{T}_n(x) - \ell, s) \geq \delta \end{array} \right\}.$$

Indeed, if an index n does not belong to $A_X(\delta, s; \ell)$, then all three good inequalities hold in X. By the continuity of L, all three good inequalities then hold in Y, so that $n \notin A_Y(\varepsilon, t; L\ell)$. Hence,

$$A_Y(\varepsilon, t; L\ell) \subseteq A_X(\delta, s; \ell).$$

Since

$$A_X(\delta, s; \ell) \in \mathcal{J}$$

and $\mathcal{J}$ is hereditary, it follows that

$$A_Y(\varepsilon, t; L\ell) \in \mathcal{J}.$$

Because $\varepsilon \in (0, 1)$ and $t > 0$ were arbitrary, we conclude that

$$\mathcal{J}_{(\mathsf{P}_Y, \mathsf{Q}_Y, \mathsf{R}_Y)}\text{-}\lim_{n \to \infty} \mathbb{T}_n(Lx) = L\ell.$$

□

Remark 6.1. *Because L is linear, neutrosophic continuity at zero implies neutrosophic continuity at every point. Indeed, for $x, y \in X$,*

$$Lx - Ly = L(x - y).$$

Thus, the behavior of L near an arbitrary point is determined by its behavior near zero.

6.2. **Inclusion Relations between Ideals.** We next show that Tribonacci ideal convergence is preserved when the underlying ideal is enlarged.

Theorem 6.2. *Let $(X, \mathsf{P}, \mathsf{Q}, \mathsf{R}, *, \diamond)$ be a non-Archimedean neutrosophic normed space, and let $\mathcal{J}_1$ and $\mathcal{J}_2$ be proper admissible ideals of $\mathbb{N}$ such that $\mathcal{J}_1 \subseteq \mathcal{J}_2$. If a sequence $x = (x_k)$ is Tribonacci $\mathcal{J}_1$-convergent to $\ell \in X$, then it is also Tribonacci $\mathcal{J}_2$-convergent to ℓ.*

Proof. Suppose that

$$\mathcal{J}_{1,(\mathsf{P},\mathsf{Q},\mathsf{R})}\text{-}\lim_{n \to \infty} \mathbb{T}_n(x) = \ell.$$

By Theorem 3.1, for every $\varepsilon \in (0, 1)$ and $t > 0$, the exceptional set

$$A_{\mathbb{T}}(\varepsilon, t; \ell) \in \mathcal{J}_1$$

Since

$$\mathcal{J}_1 \subseteq \mathcal{J}_2,$$

we also have

$$A_{\mathbb{T}}(\varepsilon, t; \ell) \in \mathcal{J}_2.$$

Therefore, again by Theorem 3.1,

$$\mathcal{J}_{2,(\mathsf{P},\mathsf{Q},\mathsf{R})}\text{-}\lim_{n\to\infty} \mathbb{T}_n(x) = \ell.$$

□

7. Valuation-Regular Tribonacci Matrices

In the classical theory of matrix summability, regularity is usually formulated with respect to the ordinary absolute value. However, when the underlying linear space is defined over a non-Archimedean valued field, the behavior of the matrix coefficients must be examined with respect to the corresponding valuation. In this section, we introduce a valuation-compatible notion of regularity for the Tribonacci matrix and establish a sufficient condition under which ordinary convergence is preserved by the Tribonacci transformation. We also provide an example showing that Tribonacci ideal convergence is, in general, strictly weaker than ordinary Tribonacci convergence.

Definition 7.1. *Let $\mathbb{T} = (a_{nk})$ be a lower triangular matrix over a non-Archimedean valued field $(\mathbb{K}, |\cdot|)$. The matrix $\mathbb{T}$ is said to be* regular with respect to the valuation $|\cdot|$ *if the following conditions are satisfied:*

(1)
$$\sum_{k=1}^{n} a_{nk} = 1$$
for every $n \in \mathbb{N}$;

(2)
$$\sup_{\substack{n\in\mathbb{N}\\ 1\le k\le n}} |a_{nk}| \le 1;$$

(3) *for every fixed $k \in \mathbb{N}$,*
$$\lim_{n\to\infty} |a_{nk}| = 0.$$

Theorem 7.1. *Let $(X, \|\cdot\|)$ be a non-Archimedean normed space over the non-Archimedean valued field $(\mathbb{K}, |\cdot|)$. Define*

$$\mathsf{P}(x,t) = \frac{t}{t+\|x\|}, \qquad \mathsf{Q}(x,t) = \frac{\|x\|}{t+\|x\|}$$

and

$$\mathsf{R}(x,t) = \frac{\|x\|}{t+\|x\|}, \qquad x \in X, \quad t > 0.$$

Let

$$a * b = \min\{a, b\}, \qquad a \diamond b = \max\{a, b\}.$$

Then $(X, \mathsf{P}, \mathsf{Q}, \mathsf{R}, *, \diamond)$ is a non-Archimedean neutrosophic normed space.

Suppose that the Tribonacci matrix $\mathbb{T} = (a_{nk})$ is regular with respect to the valuation $|\cdot|$. If a sequence $x = (x_k)$ satisfies

$$\lim_{k\to\infty} \|x_k - \ell\| = 0$$

for some $\ell \in X$, then

$$\lim_{n\to\infty} \|\mathbb{T}_n(x) - \ell\| = 0.$$

Consequently, x is Tribonacci $\mathcal{J}$-convergent to ℓ for every proper admissible ideal $\mathcal{J}$ of $\mathbb{N}$.

Proof. Since $\mathbb{T}$ is regular with respect to the valuation, we have

$$\sum_{k=1}^{n} a_{nk} = 1$$

for every $n \in \mathbb{N}$. Therefore,

$$\begin{aligned}\mathbb{T}_n(x) - \ell &= \sum_{k=1}^{n} a_{nk}x_k - \left(\sum_{k=1}^{n} a_{nk}\right)\ell \\ &= \sum_{k=1}^{n} a_{nk}(x_k - \ell).\end{aligned}$$

Let $\eta > 0$ be arbitrary. Since

$$\|x_k - \ell\| \longrightarrow 0,$$

there exists $k_0 \in \mathbb{N}$ such that

$$\|x_k - \ell\| < \eta$$

for every $k > k_0$.

For $n > k_0$, the non-Archimedean triangle inequality and the homogeneity of the norm give

$$\begin{aligned}\|\mathbb{T}_n(x) - \ell\| &= \left\|\sum_{k=1}^{n} a_{nk}(x_k - \ell)\right\| \\ &\leq \max_{1\leq k\leq n}\{\|a_{nk}(x_k - \ell)\|\} \\ &= \max_{1\leq k\leq n}\{|a_{nk}|\|x_k - \ell\|\} \\ &\leq \max\left\{\max_{1\leq k\leq k_0}|a_{nk}|\|x_k - \ell\|, \max_{k_0<k\leq n}|a_{nk}|\|x_k - \ell\|\right\}.\end{aligned}$$

For every $k > k_0$, Definition 7.1 gives

$$|a_{nk}| \leq 1.$$

Hence,

$$|a_{nk}|\|x_k - \ell\| \leq \|x_k - \ell\| < \eta.$$

It follows that

$$\max_{k_0<k\leq n}|a_{nk}|\|x_k - \ell\| < \eta.$$

For each fixed $k \in \{1, 2, \ldots, k_0\}$, we have

$$|a_{nk}| \longrightarrow 0 \qquad \text{as } n \to \infty.$$

Therefore, there exists $n_k \in \mathbb{N}$ such that

$$|a_{nk}|\|x_k - \ell\| < \eta$$

whenever $n \geq n_k$. Since there are only finitely many indices $1 \leq k \leq k_0$, let

$$n_0 = \max\{k_0 + 1, n_1, n_2, \ldots, n_{k_0}\}.$$

Then, for every $n \geq n_0$,

$$\max_{1\leq k\leq k_0}|a_{nk}|\|x_k - \ell\| < \eta.$$

Combining the two estimates, we obtain

$$\|\mathbb{T}_n(x) - \ell\| < \eta$$

for every $n \geq n_0$. Since $\eta > 0$ was arbitrary,

$$\lim_{n\to\infty}\|\mathbb{T}_n(x) - \ell\| = 0.$$

Thus,

$$\mathbb{T}_n(x) \longrightarrow \ell$$

in the non-Archimedean norm. Consequently,

$$\mathsf{P}(\mathbb{T}_n(x) - \ell, t) \longrightarrow 1,$$
$$\mathsf{Q}(\mathbb{T}_n(x) - \ell, t) \longrightarrow 0$$

and

$$\mathsf{R}(\mathbb{T}_n(x) - \ell, t) \longrightarrow 0$$

for every $t > 0$.

In particular, x is ordinarily Tribonacci-convergent to ℓ. Taking $M = \mathbb{N} \in \mathcal{F}(\mathcal{J})$, it follows that x is Tribonacci $\mathcal{J}^*$-convergent to ℓ. Therefore, by Theorem 4.1,

$$\mathcal{J}_{(\mathsf{P},\mathsf{Q},\mathsf{R})}\text{-}\lim_{n\to\infty} \mathbb{T}_n(x) = \ell.$$

□

Remark 7.1. *The valuation regularity conditions in Definition 7.1 must be verified with respect to the valuation of the underlying field. In particular, regularity of the Tribonacci matrix with respect to the ordinary absolute value does not automatically imply regularity with respect to a non-Archimedean valuation.*

Example 7.1. *Let*

$$\mathcal{J}_d = \{A \subseteq \mathbb{N} : d(A) = 0\},$$

where

$$d(A) = \lim_{n\to\infty} \frac{1}{n} |A \cap \{1, 2, \ldots, n\}|$$

whenever the limit exists. Thus, $\mathcal{J}_d$ is the ideal of subsets of $\mathbb{N}$ having natural density zero.

Let $u \in X$ be a fixed nonzero vector, and define a sequence $z = (z_n)$ by

$$z_n = \begin{cases} u, & n = m^2 \text{ for some } m \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

Since the Tribonacci matrix is lower triangular and its diagonal entries satisfy

$$a_{nn} \neq 0,$$

there exists a unique sequence $x = (x_n)$ such that

$$\mathbb{T}_n(x) = z_n$$

for every $n \in \mathbb{N}$. Indeed, the sequence may be constructed recursively by

$$x_n = a_{nn}^{-1} \left(z_n - \sum_{k=1}^{n-1} a_{nk} x_k \right).$$

Let $\varepsilon \in (0, 1)$ and $t > 0$. If n is not a perfect square, then

$$\mathbb{T}_n(x) = z_n = 0.$$

Hence,

$$\mathsf{P}(\mathbb{T}_n(x), t) = 1, \qquad \mathsf{Q}(\mathbb{T}_n(x), t) = 0$$

and

$$\mathsf{R}(\mathbb{T}_n(x), t) = 0.$$

Therefore, the exceptional set corresponding to the limit zero satisfies

$$A_{\mathbb{T}}(\varepsilon, t; 0) \subseteq \{m^2 : m \in \mathbb{N}\}.$$

Since the set of perfect squares has natural density zero,

$$\{m^2 : m \in \mathbb{N}\} \in \mathcal{J}_d.$$

Because $\mathcal{J}_d$ is hereditary, we obtain

$$A_{\mathbb{T}}(\varepsilon, t; 0) \in \mathcal{J}_d.$$

Thus,

$$\mathcal{J}_{d,(\mathsf{P},\mathsf{Q},\mathsf{R})}\text{-}\lim_{n\to\infty} \mathbb{T}_n(x) = 0.$$

On the other hand, since $u \neq 0$, the definiteness axioms of the neutrosophic norm imply that, for every fixed $t_0 > 0$,

$$\mathsf{P}(u, t_0) < 1, \qquad \mathsf{Q}(u, t_0) > 0$$

and

$$\mathsf{R}(u, t_0) > 0.$$

Hence, one may choose $\varepsilon_0 \in (0, 1)$ such that at least one of the following inequalities holds:

$$\mathsf{P}(u, t_0) \leq 1 - \varepsilon_0,$$

$$\mathsf{Q}(u, t_0) \geq \varepsilon_0$$

or

$$\mathsf{R}(u, t_0) \geq \varepsilon_0.$$

Since

$$\mathbb{T}_{m^2}(x) = z_{m^2} = u$$

for every $m \in \mathbb{N}$, there are infinitely many indices at which the transformed sequence fails to satisfy the ordinary convergence conditions for the limit zero. Therefore,

$$\mathbb{T}_n(x) \not\longrightarrow 0$$

in the ordinary neutrosophic sense.

Consequently, Tribonacci $\mathcal{J}_d$-convergence is strictly weaker than ordinary Tribonacci convergence.

8. Discussion and Suggestions for Future Developments

The present study develops a unified framework for Tribonacci ideal convergence in non-Archimedean neutrosophic normed spaces. It combines three distinct mathematical structures: the set-theoretic flexibility of ideal convergence, the summability properties of the Tribonacci matrix, and the three-component geometry of a neutrosophic norm. This combination extends both the ideal-convergence theory initiated in [35] and the theory of convergence in neutrosophic normed spaces developed in [26, 28].

8.1. **Discussion of the Main Findings.** The proposed convergence method is formulated through three exceptional index sets corresponding to the truth-membership, indeterminacy-membership, and falsity-membership functions. This componentwise formulation is important because convergence in a neutrosophic normed space requires the simultaneous conditions

$$\Omega(\mathcal{T}_n(x) - \ell, t) \longrightarrow 1,$$

$$\Upsilon(\mathcal{T}_n(x) - \ell, t) \longrightarrow 0$$

and

$$\Theta(\mathcal{T}_n(x) - \ell, t) \longrightarrow 0.$$

The use of an admissible ideal allows these conditions to fail on a prescribed negligible subset of $\mathbb{N}$. Consequently, the proposed notion includes ordinary Tribonacci convergence and several density-related convergence methods as particular cases.

The equivalent characterizations obtained in this paper show that Tribonacci $\mathcal{I}$-convergence may be expressed in terms of a single exceptional set, three componentwise exceptional sets, good-index sets belonging to the associated filter, or ideal limits of the three neutrosophic components. These characterizations provide different tools for subsequent applications. The

exceptional-set formulation is suitable for direct verification, whereas the filter formulation is particularly effective in proving uniqueness, linearity, and preservation results.

The uniqueness theorem confirms that the proposed method determines at most one Tribonacci $\mathcal{I}$-limit. This result is not merely a formal extension of uniqueness in an ordinary normed space. Its proof depends on the interaction between the non-Archimedean triangle properties, the continuous t-norm and t-conorm, and the filter associated with the ideal. The result therefore demonstrates that the three-component neutrosophic structure remains compatible with a well-defined generalized limit.

The linearity results further show that the class of Tribonacci $\mathcal{I}$-convergent sequences is stable under finite linear combinations. In particular, if

$$\mathcal{I}_{(\Omega,\Upsilon,\Theta)}\text{-}\lim_{n\to\infty}\mathcal{T}_n(x)=\ell_1$$

and

$$\mathcal{I}_{(\Omega,\Upsilon,\Theta)}\text{-}\lim_{n\to\infty}\mathcal{T}_n(y)=\ell_2,$$

then, for every $\alpha,\beta\in\mathbb{K}$,

$$\mathcal{I}_{(\Omega,\Upsilon,\Theta)}\text{-}\lim_{n\to\infty}\mathcal{T}_n(\alpha x+\beta y)=\alpha\ell_1+\beta\ell_2.$$

Thus, the proposed convergence method respects the algebraic structure of the underlying vector space.

The relationship between Tribonacci $\mathcal{I}$-convergence and Tribonacci $\mathcal{I}^*$-convergence is another significant aspect of the present work. Tribonacci $\mathcal{I}^*$-convergence requires ordinary neutrosophic convergence along an index set belonging to $\mathcal{F}(\mathcal{I})$. This condition implies Tribonacci $\mathcal{I}$-convergence for every proper admissible ideal. Conversely, property (AP) permits the construction of a sufficiently large subsequence on which ordinary convergence occurs. The role of property (AP) is consistent with its importance in the general theory of ideal-Cauchy and ideal-star convergence [40].

The Tribonacci $\mathcal{I}$-Cauchy criterion developed in this study provides a connection between generalized convergence and completeness. Every Tribonacci $\mathcal{I}$-convergent sequence is Tribonacci $\mathcal{I}$-Cauchy. Under completeness of the underlying non-Archimedean neutrosophic normed space and property (AP) of the ideal, the converse also holds. This result gives an intrinsic criterion for convergence without requiring the limit to be known in advance.

The operator-preservation theorem shows that neutrosophically continuous linear operators preserve Tribonacci ideal limits. Such a property is essential for further operator-theoretic developments, since it allows the convergence behavior of a sequence to be transferred between different non-Archimedean neutrosophic normed spaces. The inclusion theorem for ideals also shows that

$$\mathcal{I}_1\subseteq\mathcal{I}_2,$$

then

$$c_{\mathcal{T}}^{\mathcal{I}_1}\subseteq c_{\mathcal{T}}^{\mathcal{I}_2}.$$

Hence, enlarging the ideal produces a weaker convergence method and, therefore, a potentially larger convergence class. Related questions concerning matrices that preserve convergence with respect to two different ideals have been investigated in the classical setting by Connor and Leonetti [42].

The present results also complement existing investigations of Tribonacci-generated sequence spaces. Dağlı and Yaying studied paranormed sequence spaces generated by a regular Tribonacci matrix, including their bases, duals, and matrix transformations [47]. Khan, Arshad, and Khan examined Tribonacci convergent sequence spaces in neutrosophic normed spaces [50]. The present work differs by incorporating both an admissible ideal and a non-Archimedean neutrosophic structure, thereby allowing sparse or ideal-small irregularities in the transformed sequence.

8.2. **Limitations of the Present Framework.** Although the results establish a basic theory, several limitations should be acknowledged. First, the investigation is based on a specific regular Tribonacci matrix. Therefore, the results do not automatically apply to an arbitrary triangular matrix or to matrices generated by other recurrence relations.

Second, the equivalence between Tribonacci $\mathcal{I}$-convergence and Tribonacci $\mathcal{I}^*$-convergence is established under property (AP). Although this condition is standard and useful, it excludes certain ideals. A complete characterization of the ideals for which the two convergence methods coincide has not been obtained here.

Third, the operator results are restricted to linear operators that are neutrosophically continuous. Nonlinear operators, compact operators, and operators satisfying weaker continuity conditions have not been considered.

Fourth, the present study primarily concerns qualitative convergence. No explicit rate of Tribonacci ideal convergence has been introduced. Similarly, Tauberian conditions that permit the recovery of ordinary convergence from Tribonacci ideal convergence remain unexplored.

Finally, the current framework deals with single sequences. Double, triple, and multidimensional sequences require suitable extensions of both the Tribonacci transformation and the underlying ideal. Such extensions may involve product ideals, multidimensional densities, or different forms of regularity.

8.3. **Suggestions for Future Developments.** The results obtained in this paper suggest several directions for further investigation.

1. General matrix transformations. A natural continuation is to replace the Tribonacci matrix $\mathcal{T}$ with a general infinite matrix

$$A = (a_{nk}).$$

One may introduce A-ideal convergence in a non-Archimedean neutrosophic normed space and determine conditions under which A preserves ideal limits. In particular, a neutrosophic analogue of $(\mathcal{I}, \mathcal{J})$-regularity could be formulated. This would extend the matrix-preservation results of Connor and Leonetti [42] to a three-component non-Archimedean setting.

2. Other recurrence-generated matrices. The Tribonacci recurrence may be replaced by Fibonacci, k-bonacci, Pell, Jacobsthal, or other recurrence sequences. Comparisons between the resulting convergence methods may reveal inclusion relations and equivalence conditions. Previous studies of Fibonacci statistical convergence [24, 48] and Tribonacci sequence spaces [47, 50] provide a basis for this direction.

3. Lacunary and weighted versions. One may define lacunary Tribonacci ideal convergence by using a lacunary sequence

$$\theta = (k_r)$$

and investigating exceptional indices within the intervals

$$I_r = (k_{r-1}, k_r].$$

Weighted, deferred, and order-α versions may also be introduced. These extensions could be compared with existing lacunary ideal methods in intuitionistic fuzzy normed spaces [19] and generalized statistical convergence in neutrosophic normed spaces [28].

4. Rough Tribonacci ideal convergence. A roughness parameter may be incorporated into the definition so that the transformed sequence is allowed to approach a rough limit set rather than a single point, following the developments of rough, rough statistical, and rough ideal convergence [60, 61, 62]. It would be useful to characterize rough Tribonacci ideal limit sets, cluster points, boundedness, convexity, and closedness in an NA-NNS, extending related investigations in neutrosophic normed spaces [63].

5. Rates of convergence. A quantitative version of Tribonacci ideal convergence may be introduced using a sequence of positive numbers (ρ_n) or a modulus function. For example, one may investigate conditions of the form

$$\frac{1}{\rho_n}\,|k \le n : \Omega(\mathcal{T}_k(x) - \ell, t) \le 1 - \varepsilon| \longrightarrow 0.$$

Such a theory would extend generalized statistical convergence methods already studied in neutrosophic normed spaces [28].

6. Approximation by positive linear operators. The proposed convergence method may be applied to Korovkin-type approximation theory. Let (L_n) be a sequence of positive linear operators on a suitable function space. One may seek a finite collection of test functions such that Tribonacci ideal convergence of $L_n(e_i)$ to e_i implies Tribonacci ideal convergence of $L_n(f)$ to f for every admissible function f. Fibonacci statistical Korovkin theorems developed by Kirişci and Karaisa [48] indicate that recurrence-generated convergence methods can be effective in approximation theory.

7. Operator theory in NA-NNSs. The theories of bounded and compact linear operators have been developed in fuzzy normed spaces [9, 58], while compact-operator theory has also been studied over non-Archimedean Banach spaces [59]. These results suggest developing a systematic operator theory for non-Archimedean neutrosophic normed spaces.

In particular, future investigations may introduce and characterize neutrosophically bounded, compact, finite-rank, and Fredholm operators between NA-NNSs. It would be useful to determine the relationships among neutrosophic continuity, boundedness, compactness, and convergence preservation. One may also study the structure of the corresponding operator spaces, including completeness, operator norms, spectra, kernels, ranges, and compact perturbations.

Another important problem is to determine whether Tribonacci $\mathcal{I}$-convergence is preserved under sequences of operators. More precisely, let

$$L_n : \mathcal{U} \longrightarrow \mathcal{V}$$

be a sequence of neutrosophically continuous linear operators. One may seek conditions under which

$$L_n \longrightarrow L$$

uniformly in an appropriate neutrosophic operator topology and

$$\mathcal{I}_{(\Omega_{\mathcal{U}},\Upsilon_{\mathcal{U}},\Theta_{\mathcal{U}})}\text{-}\lim_{k\to\infty} \mathcal{T}_k(x) = \ell$$

imply

$$\mathcal{I}_{(\Omega_{\mathcal{V}},\Upsilon_{\mathcal{V}},\Theta_{\mathcal{V}})}\text{-}\lim_{k\to\infty} L\big(\mathcal{T}_k(x)\big) = L(\ell).$$

It would also be worthwhile to examine preservation under compositions. For neutrosophically continuous operators

$$L : \mathcal{U} \longrightarrow \mathcal{V} \qquad \text{and} \qquad S : \mathcal{V} \longrightarrow \mathcal{W},$$

one may investigate whether the composition

$$S \circ L : \mathcal{U} \longrightarrow \mathcal{W}$$

preserves Tribonacci ideal limits and ideal-Cauchy sequences. Related matrix-preservation problems for convergence with respect to two different ideals have been studied in [42]. Extending these ideas to operator sequences in NA-NNSs could lead to suitable notions of Tribonacci $\mathcal{I}$-compactness, operator convergence, spectral theory, and Fredholm theory in the neutrosophic non-Archimedean setting.

8. Fixed-point results. Fixed-point theorems for contractive mappings have been extensively studied in non-Archimedean fuzzy metric spaces and in several neutrosophic metric frameworks

[55, 56, 57]. Moreover, Tribonacci convergence and Tribonacci Cauchy sequences in neutrosophic normed spaces have already been investigated in [50]. These developments suggest studying fixed-point problems in the present Tribonacci ideal-convergence framework.

More precisely, let $T : \mathcal{U} \to \mathcal{U}$ be a contraction or a generalized contraction, and consider the Picard iteration

$$x_{n+1} = Tx_n, \qquad n \in \mathbb{N}.$$

One may investigate conditions under which the transformed orbit

$$\mathcal{T}_n(x) = \sum_{k=1}^{n} \tau_{nk} x_k$$

is Tribonacci $\mathcal{I}$-Cauchy and consequently Tribonacci $\mathcal{I}$-convergent to a fixed point of T. It would also be interesting to establish existence and uniqueness theorems when the contractive condition or the iterative convergence fails only on a set belonging to the ideal $\mathcal{I}$. Such results could provide fixed-point principles for iterative processes affected by sparse errors, exceptional iterations, or ideal-small perturbations.

9. Double and multiple sequences. The theory may be extended to double sequences

$$x = (x_{jk})$$

by defining an appropriate two-dimensional Tribonacci transformation and using ideals on $\mathbb{N}\times\mathbb{N}$. Questions concerning Pringsheim convergence, regular convergence, double Cauchy conditions, and iterated limits would then arise. Earlier investigations of ideal convergence for double sequences in intuitionistic fuzzy normed spaces [18] may serve as a useful starting point.

10. Characterization of completeness and property (AP)**.** The relationship among ideal convergence, ideal-Cauchy conditions, completeness, and property (AP) has played an important role in the general theory of convergence through ideals [54, 40]. In the present setting, it would be worthwhile to determine whether the completeness of the underlying non-Archimedean neutrosophic normed space and the assumption that $\mathcal{I}$ satisfies property (AP) can be replaced by weaker conditions. In particular, one may investigate whether the Tribonacci $\mathcal{I}$-Cauchy criterion remains valid under sequential completeness, Tribonacci completeness, or completeness restricted to suitable transformed subsequences. It would also be useful to construct explicit counterexamples in incomplete spaces and for admissible ideals that do not satisfy property (AP). Such examples would clarify whether these hypotheses are necessary or merely sufficient for the equivalence between Tribonacci $\mathcal{I}$-convergence, Tribonacci $\mathcal{I}^*$-convergence, and the corresponding ideal-Cauchy condition.

11. Sequence-space structure. One may define

$$c_{\mathcal{T}}^{\mathcal{I}} x : x \text{ is Tribonacci } \mathcal{I}\text{-convergent}$$

and

$$c_{0,\mathcal{T}}^{\mathcal{I}} x : \mathcal{I}_{(\Omega,\Upsilon,\Theta)}\text{-}\lim_{n\to\infty} \mathcal{T}_n(x) = 0.$$

The algebraic, topological, and geometric properties of these spaces could then be investigated. Possible questions include completeness, solidity, monotonicity, sequence algebras, Schauder bases, dual spaces, and matrix transformations. Similar structural questions for regular Tribonacci matrix domains were considered in [47].

12. Applications to uncertain and hierarchical data. Neutrosophic normed spaces provide a three-component description of truth, indeterminacy, and falsity, while ideal convergence allows exceptional observations to be controlled through a prescribed admissible ideal [26, 51]. In parallel, non-Archimedean and ultrametric structures provide a natural mathematical representation of nested relationships and have been successfully employed in hierarchical clustering and high-dimensional data analysis [52, 53]. These developments suggest that the

proposed Tribonacci ideal-convergence framework may be adapted to data containing uncertainty, incomplete information, sparse irregularities, or hierarchical dependence. Nevertheless, this remains a prospective application. For each specific problem, the practical meanings of the truth-, indeterminacy-, and falsity-membership components, the underlying non-Archimedean valuation, and the choice of the ideal must first be clearly established.

8.4. **Concluding Perspective.** The theory developed in this paper provides a foundation for the study of ideal-based matrix convergence in non-Archimedean neutrosophic settings. The results show that the classical properties of a meaningful convergence method uniqueness, linearity, Cauchy characterization, subsequence representation, and preservation under continuous operators can be retained while permitting exceptional behavior controlled by an admissible ideal.

At the same time, the interaction among matrix summability, non-Archimedean geometry, and neutrosophic uncertainty produces many open questions. General matrix transformations, approximation theorems, rough convergence, multidimensional sequences, operator theory, and fixed-point applications appear to be particularly promising directions. Accordingly, the present framework should be regarded not as a terminal generalization, but as a starting point for a broader theory of ideal-based convergence in spaces with multicomponent uncertainty.

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Mukhtar Ahmad, Florentin Smarandache. **Generalized Functional Analysis in Fuzzy, Intuitionistic Fuzzy, and Neutrosophic Spaces. Theory, Convergence, Compactness, and Applications.** Neutrosophic Science International Association (NSIA) Publishing House, Gallup (NM, USA) & Guayaquil (Ecuador), 2026.